1.-Thomson's lamp revisited

INTRODUCTION

1 Although Benacerraf's criticism of Thomson's lamp argument is well founded (see below), it is far from being complete. As we will see here, it is possible to consider a new line of argument, which Benacerraf only incidentally considered, based on the formal definition of the lamp. As we will see here, that line of argument leads to a contradictory result that compromises the formal consistency of the $\omega-$order involved in all supertasks.

2 To perform an $\omega-$supertask (supertask hereafter) means to perform an $\omega-$ordered sequence of actions (tasks) in a finite interval of time. Supertasks are useful theoretical devices for the philosophy of mathematics, particularly for formal discussions on certain problems related to infinity.[1] Although their physical possibilities and implications have also been discussed[2].

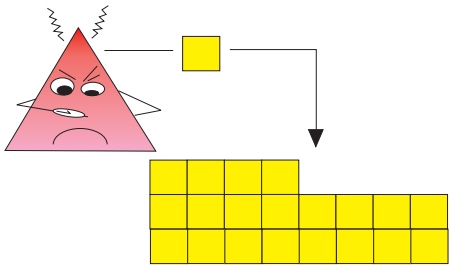

Figure 1.1: God performing Gregory's supertask.

3 Probably Gregory was the first to propose how a supertask could

[1] [29], [7], [8], [24], [5], [31], [24]

[2] [19], [20], [24], [26], [13], [15], [14], [20], [21], [22], [12], [23], [18], [2], [3], [25] [31], [16], [10], [11], [18], [9], [27]

be accomplished ([17], p. 53):

> If God can endlessly add a cubic foot to a stone -which He can- then He can create an infinitely big stone. For He need only add one cubic foot at some time, another half an hour later, another a quarter of an hour later than that, and so on *ad infinitum*. He would then have before Him an infinite stone at the end of the hour.

But the term "*supertask*" was introduced by J. F. Thomson in his seminal paper of 1954 [29]. Thomson's paper was motivated by Black's argument [6] on the impossibility of performing infinitely many successive actions and the rejections of Black's argument by R. Taylor [28] and J. Watling [30]. In his paper Thomson tried to prove the impossibility of supertasks. Thomson argument was, in turn, criticized in another seminal paper, in this case by P. Benacerraf [4]. Benacerraf's successful criticism finally motivated the foundation of a new infinitist theory: supertask theory.

4 The basic idea of Benacerraf's criticism on Thomson's argument is the impossibility to derive formal conclusions on the final state of the supermachine that performs the supertask from the sequence of states the machine traverses as a consequence of performing the supertask. But, as we will see, Benacerraf's analysis of Thomson's lamp argument is incomplete. In fact, if supertasks do not change the nature of the world, Thomson's argument can be reoriented towards the formal definition of the (theoretical) machine that performs the supertask. A definition that does not depend on the number of performed tasks; a definition that holds before, during and after performing a supertask.

THOMSON'S LAMP

5 As Thomson did in 1954, in the following discussion we will deal with one of those:

> ... reading-lamps that have a button in the base. If the lamp is *off* and you press the button the lamp goes *on*, and if the lamp

> is *on* and you press the button the lamp goes *off*. ([29], p. 5).

Let us complete Thomson's definition by explicitly declaring the following two conditions on the lamp functioning:

1. The state of the lamp (on/off) changes if, and only if, its button is pressed down.

2. The button clicking and the corresponding change of state (on/off) are both simultaneous events.

6 Assume now the lamp button is clicked at each of the infinitely many successive instants t_i, and only at them, of a strictly increasing $\omega-$ordered sequence of instants $\langle t_n \rangle_{n\in\mathbb{N}}$ defined within an finite interval of time (t_a, t_b), being t_b the limit of the sequence $\langle t_n \rangle_{n\in\mathbb{N}}$. In these conditions, at instant t_b the button of the lamp will have undergone an $\omega-$ordered sequence $\langle c_n \rangle_{n\in\mathbb{N}}$ of clicks and, consequently, the lamp's state will have changed an $\omega-$ordered infinitude of times. Or in other words, at t_b Thomson's supertask will have been completed.

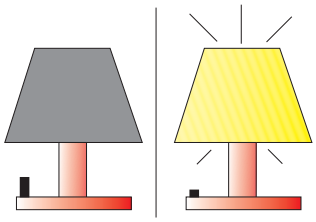

Figure 1.2: Thomson lamp off (left) and on (right).

7 Thomson tried to derive a contradiction from his supertask by speculating on the final state of the lamp at instant t_b in terms of the sequence of switchings completed along the supertask ([29], p. 5):

> [The lamp] cannot be *on*, because I did not ever turn it *on* without at once turning it *off*. It cannot be *off*, because I did in the first place turn it *on*, and thereafter I never turned *off* without at once turning it *on*. But the lamp must be either *on* or *off*. This is a contradiction.

8 It is worth noting that Thomson based his argument on the sequence of actions carried out on the lamp: it was never turned on without turning it off after, and viceversa. What Thomson tried to do is to derive the lamp's final state, the lamp's state at t_b, from the successive state changes the lamp underwent during the supertask: The

reason why the lamp cannot be *on* is because it was always turned *off* after turning it *on*. And for the same reason it cannot be *off* either. This way of arguing was criticized by Benacerraf

9 Benacerraf argued against Thomson's argument as follows ([4], p. 768):

> The only reasons Thomson gives for supposing that his lamp will not be *off* at t_b are ones which hold only for times *before* t_b. The explanation is quite simply that Thomson's instructions do not cover the state of the lamp at t_b, although they do tell us what will be its state at every instant *between* t_a and t_b (including t_a). Certainly, the lamp must be *on* or *off* (provided that it hasn't gone up in a metaphysical puff of smoke in the interval), but nothing we are told implies which it is to be. The arguments to the effect that it can't be either just have no bearing on the case. To suppose that they *do* is to suppose that a description of the physical state of the lamp at t_b (with respect to the property of being *on* or *off*) is a *logical* consequence of a description of its state (with respect to the same property) at times prior to t_b.(t_a and t_b appears respectively as t_0 and t_1 in Benacerraf's paper).

10 In short, according to Benacerraf, the problem posed by Thomson is not sufficiently described since no constraints have been placed on what happens at t_b [1]. But the only constriction on what happens at t_b is that Thomson's lamp continue to be Thomson's lamp. Or in other words, that the execution of a supertask does not change the formal definitions of the involved theoretical artifacts.

11 Consider the instant t_b, the limit of the sequence $\langle t_n \rangle_{n\in\mathbb{N}}$ of instants at which the successive clicks $\langle c_n \rangle_{n\in\mathbb{N}}$ have been performed. That instant is, therefore, the first instant *after completing* the sequence of switchings. The first instant at which the button of the lamp is no longer clicked. Let now S_b be the state of the lamp at instant t_b. Being the state of a Thomson lamp, it can only be either *on* or *off*. And this conclusion has nothing to do with the number of previously performed switchings. The lamp will be *on* or *off* because, being a Thomson's lamp, it has only two states: *on* and *off*.

12 Some infinitist claim, however, that at t_b, after performing Thomson's supertask, the lamp could be in any unknown state, even in an exotic one. But a lamp that can be in an unknown state is not a Thomson's lamp: the only possible states of a Thomson's lamp are *on* and *off*. No other alternative is possible without violating the legitime formal definition of Thomson's lamp. None.

13 Most of the infinitists claim the state S_b is the consequence of completing the $\omega-$ordered sequence of clicks $\langle c_n\rangle_{n\in\mathbb{N}}$. But to complete the sequence of clicks $\langle c_n\rangle_{n\in\mathbb{N}}$ means to perform each and every one of the infinitely many successive clicks c_i, and only them. The problem is that no click c_i of $\langle c_n\rangle_{n\in\mathbb{N}}$ originates S_b. Indeed, if c_v is any element of $\langle c_n\rangle_{n\in\mathbb{N}}$ it cannot originates S_b because in such a case only a finite number v of clicks would have been performed. Or in other terms, if we remove from $\langle c_n\rangle_{n\in\mathbb{N}}$ all clicks that don't originates S_b then all of them would be removed. All of them.

14 In those conditions, how can it be claimed that the completion of the sequence $\langle c_n\rangle_{n\in\mathbb{N}}$, *none* of whose elements originates S_b, originates just S_b? Is the completion of the sequence an additional click different from all elements of $\langle c_n\rangle_{n\in\mathbb{N}}$? If that were the case the sequence of performed clicks would be $(\omega+1)$-ordered in the place of $\omega-$ordered, but ω-supertasks are $\omega-$ordered not $(\omega+1)$-ordered.

15 At this point some infinitists claim the lamp could be at S_b by reasons unknown. But, once again, that claim violates the definition of the lamp: the state of a Thomson's lamp changes exclusively by pressing down its button, by clicking its button. So a lamp that changes its state by reasons unknown is not, by definition, a Thomson's lamp.

16 In any case, the relevant question on the state S_b is: at which instant it is originated? It is immediate to prove that S_b can only be originated at the precise instant t_b. In fact assume it is originated at an instant t before t_b. Since t_b is the limit of the sequence $\langle t_n\rangle_{n\in\mathbb{N}}$, we will have:

$$\exists v:\ t<t_n,\ \forall n\geq v \tag{1}$$

which means that at t only a finite number v of clicks have been carried out. Therefore, S_b can only be originated at t_b. But t_b is not the instant at which the sequence of switchings is completed; t_b is the first instant *after completing* the sequence. There is not, in fact, an instant at which that sequence is completed because that sequence is $\omega-$ordered and $\omega-$ordered sequences have not last element. At t_b the sequence $\langle C_n\rangle_{n\in\mathbb{N}}$ of clicks and then sequence $\langle S_n\rangle_{n\in\mathbb{N}}$ of state changes *have already been completed*. At t_b no click is performed. At t_b nothing happens that can alter the state of the lamp.

17 In accordance with 16, it cannot be claimed that S_b results from completing the sequence $\langle C_n\rangle_{n\in\mathbb{N}}$ of clicks: S_b originates at t_b and at t_b the sequence of clicks has already been completed; t_b is *posterior* to the completion of the sequence $\langle C_n\rangle_{n\in\mathbb{N}}$ of clicks. S_b can only be originated at t_b but nothing happens at t_b that can originate S_b: at t_b the button of the lamp is not pressed; at t_b the sequence of clicks has already been completed. S_b is then an impossible formal consequence derived from the assumption that it is possible to complete an uncompletable sequence of actions, uncompletable in the sense that no last element completes the sequence.

18 The fact that two incompletable sequences can be paired off by a one by one correspondence, as in the case of the above sequences of clicks and of instants, does not prove both sequences can be regarded as complete totalities.

19 At this point, all that one can expect from infinitists is to be declared incompetent to understand the meaning of the sentence: '*the state of the lamp at t_b is the result of completing the $\omega-$ordered sequence $\langle C_n\rangle_{n\in\mathbb{N}}$ of clicks, a result that manifests for the first time just at t_b*'. But, wait a moment, is not S_b the result of a click, of a pressing down the button of the lamp? Don't forget that Thomson's lamp can only change its state if you press down its button, if you click it. Furthermore, according to its formal definition, Thomson's lamp changes its state at the same instant you click its button. So if S_b appears for the first time at t_b how it is possible that it appears without clicking its button? No answer.

20 In short, S_b must of necessity be originated just at instant t_b, otherwise only a finite number if clicks would have been performed, according to 16. But, on the other hand, it cannot be originated at t_b because at t_b:

1. The sequence $\langle c_n \rangle_{n \in \mathbb{N}}$ of clicks *has already been completed*: t_b is the first instant *posterior* to that completion.

2. The button of the lamp has not been pressed neither at t_b nor after t_b.

S_b could only be, therefore, the impossible last state of an $\omega-$ordered sequence of states. The consequence of assuming the hypothesis of the actual infinity from which derives the existence of $\omega-$ordered sequences as *complete totalities* in spite of the fact that no last element completes them.

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