1.-The next rational

INTRODUCTION

1 The set $\mathbb{Q}$ of rational numbers, in its natural ordering, is densely ordered: between any two rationals infinitely many different rationals do exist. But, being denumerable [1], it can also be $\omega-$ordered: between any two successive rationals no other rational exists. The argument that follows takes advantage of this *rational singularity*.

DISCUSSION

2 For the sake of simplicity, we will deal with the set $\mathbb{Q}^+$ of positive rationals, which is also denumerable and densely ordered. Let then f be a one to one correspondence between the set $\mathbb{N}$ of natural numbers and the set $\mathbb{Q}^+$. Evidently, f induces an $\omega-$order in $\mathbb{Q}^+$ so that the set of all positive rational numbers can be written as $\{q_1, q_2, q_3, \dots\}$, being $q_i = f(i)$, $\forall i \in \mathbb{N}$

3 Let now x be a rational variable whose initial value is 1 and consider the following $\omega-$ordered sequence of definitions:

$$\left.\begin{array}{r} d_i = |q_{i+1} - q_1| \\ d_i < x \Rightarrow x = d_i \end{array}\right\} \; i = 1, 2, 3, \dots \tag{1}$$

where $|q_{i+1} - q_1|$ is the absolute value of $q_{i+1} - q_1$, and '$<$' stands for the usual dense ordering of $\mathbb{Q}$; i.e $d_i < x$ means $d_i - x < 0$. Definition

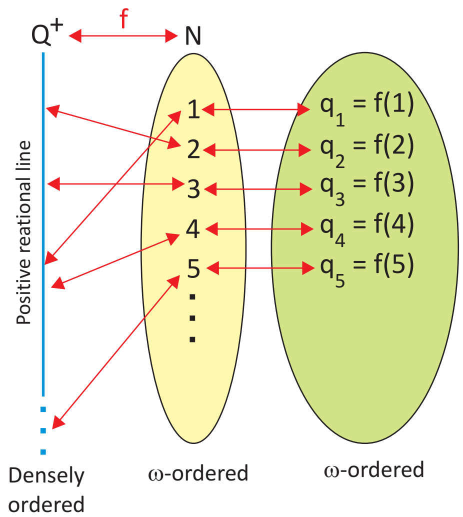

Figure 1.1: ω -Ordering the positive rational line.

(1) defines the sequence $\langle d_i \rangle_{i \in \mathbb{N}}$ of positive rationals and redefines the variable x as the defined term d_i each time the defined term d_i is less than x. Finally, we will impose to definitions (1) the following:

> Restriction 3.-*Each successive definition (1) will be carried out if, and only if, x results defined as a positive rational number.*

4 By induction, it is immediate to prove that for each natural number v, the first v definitions (1) can be carried out. Evidently the first definition (1) can be carried out since x is defined as $|q_2 - q_1|$ if $|q_2 - q_1| < 1$ or as 1 if it is not, being the result of each alternative a positive rational number. Assume that, being n any natural number, the first n definitions (1) can be carried out, which means they define x as a positive rational number. Since $|q_{n+2} - q_1|$ is a well defined positive rational number it will be, or not, less than the current rational value of x; consequently x will be defined by the $(n+1)$th definition (1) as $|q_{n+2} - q_1|$ if this number is less than its current value; if not, it will continue with its current rational value. Thus, the $(n+1)$th definition can also be carried out. We have just proved that the first definition (1) can be carried out, and that if, for any natural number n, the first n definitions (1) can be carried out, then the $(n+1)$th definition (1) can also be carried out. In consequence, for any natural number v, the first v definitions (1) can be carried out.

5 Assume that while the successive definitions (1) can be carried out, they are carried out. We will prove that once performed all possible[1] definitions (1), the rational $q_1 + x$ is the less rational greater than q_1. In effect, assume it is not. In such a case there would be a positive rational q_v greater than q_1 and less that $q_1 + x$:

$$q_1 < q_v < q_1 + x \tag{2}$$

and then, by subtracting q_1:

$$0 < q_v - q_1 < x \tag{3}$$

which is impossible because:

1. The index v of q_v is a natural number.
2. In accordance with 4 the first v definitions (1) can be carried out.
3. All possible definitions (1) have been carried out.
4. So, at least the first v definitions (1) have been carried out.
5. As a consequence of the $(v - 1)$th definition (1) we can confirm that $x \leq q_v - q_1$.
6. It is then impossible that $q_v - q_1 < x$.

In consequence our initial hypothesis must be false and $q_1 + x$ is the less rational greater than q_1.

[1]Note that if it were impossible to perform all possible definitions (1) we would be in the face of the elementary contradiction of an impossible possibility

6 It is immediate to prove, however, that once performed all possible definitions (1), the rational $q_1 + x$ is not the less rational greater than q_1. In fact, whatsoever be the value of x once performed all possible definitions (1), the rational $q_1 + 0.1x$, for instance, is greater than q_1 and less then $q_1 + x$

7 It could be argued that x results undefined because the sequence of performed definitions (1) is infinite, without a last definition completing the sequence. The same *indefinition* could therefore be expected from any other definition or procedure involving an infinite sequence of steps without a last completing step, as in the case, for instance, of all $\omega-$ordered recursive definitions. In our case, once performed all possible definitions (1), the sequence $\langle d_n \rangle_{n \in \mathbb{N}}$ would also be undefined.

Bibliography

[1] Georg Cantor, *Über eine eigenschaft aller reallen algebraishen zahlen*, Journal für die reine und angewandte Mathematik **77** (1874), 258–262.