1.-Spacetime divisibility

The less infinite ordinal

1 The first transfinite ordinal[1] ω is the less ordinal greater than all finite ordinals. It defines a type of well order called $\omega-$order:[2] a set or sequence is $\omega-$ordered if it has a first element and every element has an immediate successor and an immediate predecessor, except the first one.[3] In consequence there is not a last element in an $\omega-$ordered set or sequence. The set $\mathbb{N}$ of natural numbers in its natural order of precedence is a well known example of $\omega-$ordered set.

2 ω^*-Order is the symmetrical reflection of $\omega-$order: a set or sequence is ω^*-ordered if it has a last element and each element has an immediate predecessor and an immediate successor, except the last one. In consequence there is not first element:

$$\overbrace{\dots t_{3^*}, t_{2^*}, t_{1^*},}^{\omega^*-order} | \overbrace{t_1, t_2, t_3, \dots}^{\omega-order} \tag{1}$$

[1]Transfinite ordinals are the ordinals of the second class according to Cantor classical terminology [4]. An ordinal of the second class is of the second kind if, as ω, it is the limit of an infinite sequence of ordinals; it is of the first kind if it is of the form $\alpha+n$, where α is an ordinal of the second class second kind and n a finite ordinal (see Chapter **??** on the actual infinity).

[2]In formal terms, a set is $\omega-$ordered if it is well ordered and its ordinal is ω.

[3]Between an element and its immediate successor no other element of the sequence exists.

where $1^*, 2^*, 3^*, \ldots$ means last, last but one, last but two, etc.

3 In accordance with the definition of $\omega-$order given in 1, every element of an $\omega-$ordered set has a finite number of predecessors and an infinite number of successors. In the case of ω^*-orderevery element of an ω^*-ordered set has finitely many successors and infinitely many predecessors. This immense asymmetry in the number of predecessors and successors (ω-asymmetry) is a well known fact, although it is usually ignored in infinitist literature, particularly in supertask literature.

SUPER-ASYMMETRICAL SUPERTASKS

4 At the beginning of the second half of the XX century, the discussions on the possibilities to perform a countable infinitude of tasks or actions in a finite interval of time (a supertask according to J. F. Thomson [13]) promoted the development of new infinitist theory: supertask theory.[4] The possibilities to perform an uncountable infinitude of actions were examined, and ruled out, by P. Clark and S. Read [5]. Supertasks have also been considered from the perspective of nonstandard analysis,[5] although the possibilities to perform an *hypertask* along an hyperreal interval of time have not been discussed, despite the fact that finite hyperreal intervals can be divided into hypercountably many successive infinitesimal intervals (hyperfinite partitions).[6] But most supertasks are ω -supertasks, i.e. $\omega-$ordered sequences of actions performed in a finite (or perceived as finite) interval of time.

5 Let $\langle t_n \rangle_{n \in \mathbb{N}}$ be any strictly increasing and $\omega-$ordered sequence of instants within the finite real interval (t_a, t_b) whose limit is t_b. And let S be a supertask whose infinitely many actions $\langle a_n \rangle_{n \in \mathbb{N}}$ are performed at the infinitely many successive instants of $\langle t_n \rangle_{n \in \mathbb{N}}$, each action a_i performed at the precise instant t_i.

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4 [3], [15], [13], [14], [2]
5 [11], [10], [1], [9]
6 [12], [6], [8], [7], etc.

6 It seems convenient to recall that the limit t_b is not the instant at which S ends[7] but the first instant after S has ended, the first instant after performing all the infinitely many actions $\langle a_n \rangle_{n\in\mathbb{N}}$. Being t_b the limit of $\langle t_n \rangle_{n\in\mathbb{N}}$, at any instant t before t_b and arbitrarily close to it, only a finite number of tasks will have been performed and infinitely many of them still remain to be performed (ω -asymmetry).

7 To grasp the colossal magnitude of the above ω-asymmetry, assume the interval $[t_a,\ t_b]$ is trillions of times greater than the age of the universe and consider an interval of time $\tau = 0.000\ldots001$ seconds so small that we would need trillions and trillions of standard pages to write all its zeroes between the decimal point and the final digit 1, a number of pages so huge that the whole visible universe[8] would not have sufficient room for them; well, only a finite number of tasks will have been preformed during the trillions of years elapsed between t_a and $t_b - \tau$ while infinitely many tasks, practically all of them, will have to be performed just in our unimaginably small interval of time τ. Thus, rather than anaesthetic, ω -asymmetry is repulsive.

8 And things can get worse. Assume we remove from $[t_a, t_b]$ all instants at which infinitely many tasks of the above supertask S still remain to be performed. We would have to remove all instants within $[t_a, t_b]$, except t_b. In fact, let t be any instant within $[t_a, t_b]$ different from t_b. Since t_b is the limit of $\langle t_n \rangle_{n\in\mathbb{N}}$, we will have:

$$\exists t_v \in \langle t_n \rangle_{n\in\mathbb{N}} :\ t < t_v \tag{2}$$

so that at t only a finite number $v - 1$ of tasks have been carried out and then infinitely many tasks still have to be performed. In consequence t has to be removed from $[t_a, t_b]$. Therefore, and being t any instant within $[t_a, t_b]$ different from t_b, all instants within $[t_a, t_b]$, except t_b, have to be removed from that interval. So at t_b, the first instant after completing the supertask, there still remain infinitely many tasks to be performed.

[7]There is not an instant at which S ends because $\langle a_n \rangle_{n\in\mathbb{N}}$ is $\omega-$ordered and $\omega-$ordered sequences have not last element.

[8]A sphere of 93000 billions light years.

SPACETIME DICHOTOMIES

9 Consider any finite interval of time $[t_a, t_b]$ and within it two sequences of instants, the $\omega-$ordered sequence of t-instants:

$$\langle t_i \rangle : \; t_i = t_a + \frac{2^i - 1}{2^i}(t_b - t_a), \; \forall i \in \mathbb{N} \tag{3}$$

and the ω^*-ordered sequence of t^*-instants:

$$\langle t^*_{i^*} \rangle : \; t^*_{i*} = t_a + \frac{1}{2^i}, \; \forall i \in \mathbb{N} \tag{4}$$

where i^* stands for the last but $(i - 1)$ element of the ω^*-ordered sequence $\langle t^*_{i^*} \rangle_{i \in \mathbb{N}}$.

10 We will examine the way the successive t^*-instants of $\langle t^*_n \rangle_{n \in \mathbb{N}}$ and the successive t-instants of $\langle t_n \rangle_{n \in \mathbb{N}}$ elapse as time passes from t_a to t_b, for this we will make use of the two following functions:

$$f^*(t) = \text{number of elapsed } t^*\text{-instants at } t, \forall t \in [t_a, t_b] \tag{5}$$

$$f(t) = \text{number t-instants still not elapsed at } t, \forall t \in [t_a, t_b] \tag{6}$$

11 In accordance with the definitions of ω^*-order and $\omega-$order we can write:

$$f^*(t) = \begin{cases} 0 \text{ if t} = t_a \\ \aleph_o \text{ if } t > t_a \end{cases} \qquad f(t) = \begin{cases} \aleph_o \text{ if } t < t_b \\ 0 \text{ if } t = t_b \end{cases} \tag{7}$$

Otherwise, if there would exist an instant t such that $f^*(t) = n$ or $f(t) = n$, being n a natural number, then there would also exist the impossible firsts n elements of an ω^*-ordered sequence, or the impossible lasts n elements of an $\omega-$ordered sequence.

12 According to 11, the functions f^* and f are well defined for every t in $[t_a, t_b]$; they map the interval $[t_a, t_b]$ onto the set of two elements $\{0, \aleph_o\}$:

$$f^* : \; [t_a, t_b] \mapsto \{0, \aleph_o\} \tag{8}$$

$$f \; : \; [t_a, t_b] \mapsto \{0, \aleph_o\} \tag{9}$$

13 The function f^* defines, therefore, a dichotomy, t^*-dichotomy:

- Regarding the number of elapsed t^*-instants when time passes from t_a to t_b only two values are possible: 0 and $\aleph_o$.

The function f also defines a dichotomy, t-dichotomy:

- Regarding the number of t-instants to elapse as time passes from t_a to t_b only two values are possible: $\aleph_o$ and 0.

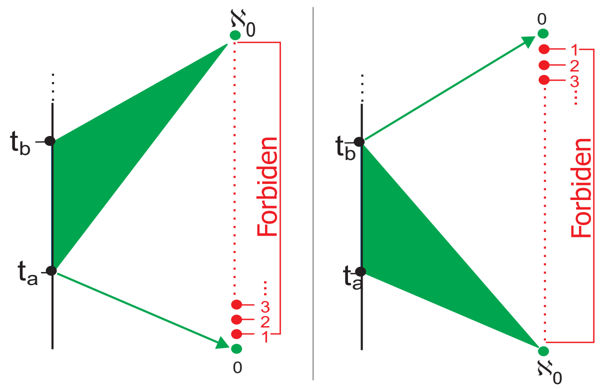

Figure 1.1: t^*-dichotomy (left) and t-dichotomy (right)

14 With respect to the number of t^*-instants elapsed from t_a, the passing of time from t_a to t_b can only exhibit two states: the state $S^*(0)$ at which no t^*-instant has elapsed, and the state $S^*(\aleph_o)$ at which infinitely many t^*-instants have already elapsed. Intermediate finite states $S^*(n)$ at which only a finite number n of t^*-instants have elapsed are not possible. The passing of time becomes $S^*(\aleph_o)$ *directly* from $S^*(0)$. Similarly, with respect to the number of t-instants not elapsed, the passing of time from t_a to t_b can only exhibit two states: $S(\aleph_o)$ and $S(0)$; without intermediate finite states $S(n)$ at which only a finite number n of t-instants still have to elapse. The passing of time reaches the state $S(0)$ *directly* from $S(\aleph_o)$.

Spacetime divisibility

15 We will now examine the duration of the transitions:

$$S^*(0) \to S^*(\aleph_o) \tag{10}$$
$$S(\aleph_o) \to S(0) \tag{11}$$

According to (7) the number of t^*-instants elapsed from t_a and the number of t-instants still not elapsed from t_a are well defined along the whole interval $[t_a, t_b]$. On the other hand, both transitions must take place within the same interval $[t_a, t_b]$.

16 Although the real interval $[t_a, t_b]$ is densely ordered, the sequences $\langle t^*_{i*}\rangle_{i\in\mathbb{N}}$ and $\langle t_i\rangle_{i\in\mathbb{N}}$ within $[t_a, t_b]$, are not. These sequences are ω^*-ordered and $\omega-$ordered respectively, which means that t^*-instants and t-instants are strictly successive, i.e. between any t^*-instant and its immediate successor no other t^*-instant exists; and the same applies to t-instants. Thus, t^*-instants and t-instants can only elapse successively, one at a time, and in such a way that between any two of those successive instants a time greater than zero always passes. In consequence, the number of t^*-instants elapsed from t_a can only increase *one by one*, from 0 to $\aleph_o$. The same applies to the way the number of t-instants still not elapsed from t_a decreases from $\aleph_o$ to 0. This successiveness will play an important role in the next discussion.

17 As a consequence of the t^*-dichotomy, the number of t^*-instants elapsed from t_a has to increase one by one from 0 to $\aleph_o$ without traversing the increasing sequence of natural numbers 1, 2, 3, ... Analogously, the number of t-instants to elapse has to decrease one by one from $\aleph_o$ to 0 without traversing the decreasing sequence of natural numbers ..., 3, 2, 1 (see Figure 1.2).

18 The duration of the transition $S^*(0) \to S^*(\aleph_o)$ is, according to 16, the interval of time within $[t_a, t_b]$ during which the number of t^*-instants elapsed from t_a increases, *one by one and with a non-zero time interval between each increase*, from zero to $\aleph_o$. Similarly, the

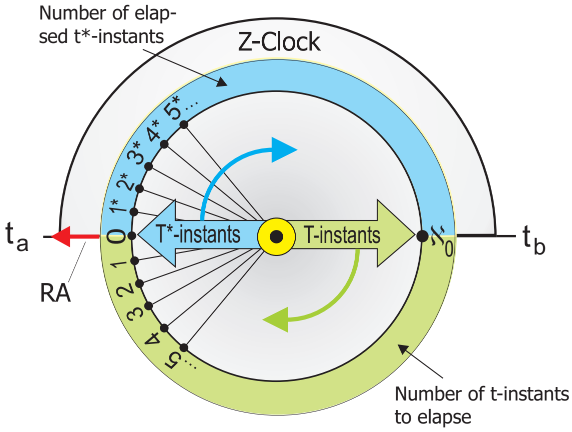

Figure 1.2: As time (red arrow RA) passes from t_a to t_b the arrow of t^*-instants will turn clockwise from 0 to $\aleph_o$ without passing over the successive radius 1, 2, 3, ... At the same time the arrow of t-instants will turn clockwise from $\aleph_o$ to 0 without passing over the successive radius . . . 3, 2, 1.

duration of the transition $S(\aleph_o) \to S(0)$ is the interval of time within $[t_a, t_b]$ during which the number of t-instants to elapse decreases, *one by one and with a non-zero time interval between each decrease*, from $\aleph_o$ to zero.

19 Since a time greater than zero always elapses between any two successive t^*-instants, a time greater than zero must necessarily elapse between an infinite number of successive t^*-instants. This is why the transition $S^*(0) \to S^*(\aleph_o)$ will necessarily take a time greater than zero. The same conclusion, and for the same reason, applies to the transition $S(\aleph_o) \to S(0)$.

20 Assume the transition $S^*(0) \to S^*(\aleph_o)$ lasts a time τ, being τ any positive real number. Let τ' be any instant within the real interval $(0, \tau)$. According to t^*-dichotomy, the number of elapsed t^*-instants at $t_a + \tau'$ is $\aleph_o$, and then the transition $S^*(0) \to S^*(\aleph_o)$ already has finished. Consequently the transition $S^*(0) \to S^*(\aleph_o)$ lasts a time less than τ. And being τ any real number greater than 0, we must conclude the duration of $S^*(0) \to S^*(\aleph_o)$ is less than any real number greater than zero. Or in other words, it has to last a null time.

21 An argument similar to 20 proves the transition $S(\aleph_o) \rightarrow S(0)$ has also to be instantaneous. It could be argued that the transition $S(\aleph_o) \rightarrow S(0)$ lasts a time t_b - t_a, but this is impossible because at $t_a + \tau$, being τ any positive real number less than $t_b - t_a$, the number of t-instants still not elapsed is $\aleph_o$, and then the transition $S(\aleph_o) \rightarrow S(0)$ has not begun. In consequence it lasts an amount of time less than $t_b - t_a$.

22 According to 20 and 21, infinitely many successive t^*-instants and infinitely many successive t-instants have to elapse simultaneously. But this is impossible because successive instants cannot elapse simultaneously: between any two of those successive instants t^*_n, t^*_{n+1} (or t_n, t_{n+1}) a finite interval of time greater than zero always passes: just the interval $[t^*_n, t^*_{n+1}]$ (or the interval $[t_n, t_{n+1}]$ in the case of t-instants). The transitions $S^*(0) \rightarrow S^*(\aleph_o)$ and $S(\aleph_o) \rightarrow S(0)$ have to last times greater than zero, but they cannot last times greater than zero (20/21). We have therefore two contradictions proving the impossibility of dividing any finite interval of time into an actual infinitude of ω^*−ordered parts and into an actual infinitude of ω−ordered parts (see Z-Clock in Figure 1.2).

23 Any countable infinite partition of time has to be α-ordered, being α an ordinal of the second class (first or second kind). Thus, we will have:

$$\alpha = \omega \tag{12}$$

or:

$$\alpha = \omega + \beta \tag{13}$$

where β is an ordinal or the second class (first or second kind). In consequence, any transfinite partition of time has to contain at least an impossible ω−ordered partition. Denumerable partitions of time are therefore impossible. And since any non countable division contains infinitely many denumerable partitions we must conclude time is not infinitely divisible.

24 If in the place of the passage of time and the sequences of t^*-instants and t-instants we were considered the uniform linear motion of a particle traversing the Z^* points $\langle z_n^* \rangle_{n\in\mathbb{N}}$ and Z-points $\langle z_n \rangle_{n\in\mathbb{N}}$ defined within the real interval $[0,1]$ of the real line as:

$$\langle z_{i^*}^* \rangle : \; z_{i^*}^* = \frac{1}{2^i}, \; \forall i \in \mathbb{N} \tag{14}$$

$$\langle z_i \rangle : \; z_i = \frac{2^i - 1}{2^i}, \; \forall i \in \mathbb{N} \tag{15}$$

We would have come to the same conclusion, and for the same reasons, on the infinite divisibility of space we have come on the infinite divisibility of time.

25 The above conclusions on the divisibility of space and time not only apply to space and time but to the very notion of densely ordered continuum.

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