

1.-Reinterpreting the paradoxes of reflexivity

INTRODUCTION

1 If after pairing each and every element of a set A with a different element of a set B all elements of B result paired, we say both sets have the same cardinality (the same number of elements). But if one or more elements of B result unpaired and B is infinite, we are not allowed to say both sets have different cardinality. In this chapter we discuss why we are not allowed to do it. As we will see, the existence of both exhaustive and non-exhaustive injections¹ between two infinite sets could be indicating they have and not have the same cardinality. Thus, the arbitrary distinction of the exhaustive injections to the detriment of the non-exhaustive ones could be concealing a fundamental contradiction in set theory.

2 Most of the paradoxes related to infinity result from the violation of the euclidian Axiom of the Whole and the Part,² among them the so called paradoxes of reflexivity in which the elements of a whole are paired off with the elements of one of its proper parts.³ Galileo's paradox⁴ is a well known example of reflexive paradox. Authors as

¹An injection is a correspondence between the elements of two sets A and B such that each and every element of A is paired off with a different element of B .

²The assumption that the whole is greater than the part is one of the common notions that appears in the first book of Euclid's Elements [8, p 19].

³[12], [7].

⁴The natural numbers can be paired with their squares [10].

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Proclus, J. Filopón, Thabit ibn Qurra al-Harani, R. Grosseteste, G. de Rimini, W. of Ockham etc. found many other examples [12].

3 The strategy of pairing off the elements of two sets is not just a modern invention. Aristotle used it to refute Zeno’s Dichotomies.⁵ And since then, it has been used by most of the authors in the field of set theory with different discursive purposes, although, before Dedekind and Cantor, they were never used (including the case of Bolzano [3]) as an instrument to consummate the violation of the old euclidian axiom.

4 Things began to change with Dedekind, who stated the definition of an infinite set just on the basis of that violation: a set is infinite if its elements can be put into a one to one correspondence with the elements of one of its proper subsets [6]. Dedekind and Cantor inaugurated the so called paradise of the actual infinity, where exhaustive injections (bijections or one to one correspondences) play a capital role.

PARADOXES OR CONTRADICTIONS?

5 As is well known, an exhaustive injection between two sets A and B is a correspondence between the elements of both sets in which each element of A is paired off with a different element of B , and all elements of A and of B result paired. When at least one element of the set B results unpaired the injection is said non-exhaustive. Exhaustive and non-exhaustive injections can be used to compare the cardinality of finite sets. But if the compared sets are infinite, only exhaustive injections are allowed. No reason has ever been given to justify that arbitrary distinction.

6 It seems reasonable to assume that if after pairing every element of a set A with a different element of a set B , all elements of B result paired, then A and B have the same number of elements. But it

⁵Aristotle finally rejected his pairing method and proposed the distinction between the actual and the potential infinity [2], [1].

seems also reasonable to assume, and for the same elementary reasons, that if after pairing every element of a set A with a different element of a set B one or more elements of the set B remain unpaired, then A and B have not the same number of elements. Notice that both exhaustive injections and non-exhaustive injections make use of *the same basic method of pairing elements*, without carrying out any finite or transfinite arithmetic operation.

7 Exhaustive and non-exhaustive injections should have the same validity as instruments to compare the cardinality of infinite sets just because they use *exactly the same comparison method*. However, only exhaustive injections do have that privilege. The problem here is that the existence of both exhaustive and non-exhaustive injections between two infinite sets could be indicating the existence of an elementary contradiction (that both infinite sets have and have not the same cardinality), in which case the distinction of exhaustive injections would be the distinction of a term of a contradiction to the detriment of the other.

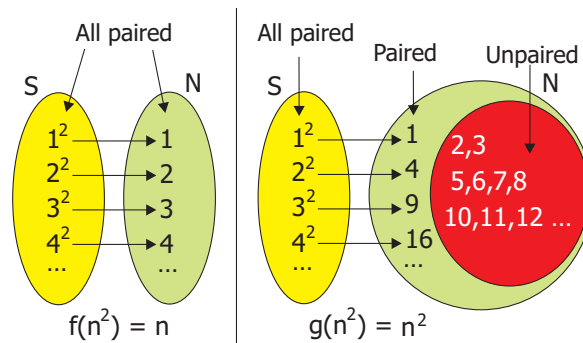


Figure 1.1: The suspicious power of the ellipsis: the sets S and N have (left) and not have (right) the same number of elements.

8 It should be recalled at this point that the existence of transfinite cardinals as \aleph_0 , for which it holds $\aleph_0 = \aleph_0 + \aleph_0$ and the like, is derived from the hypothetical existence (Axiom of Infinity) of infinite sets, i.e. of sets whose elements can, by definition, be paired with the ele-

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ments of some of their proper subsets. So, under penalty of circular reasoning, we cannot infer from the deduced existence of transfinite cardinals with arithmetical peculiarities (that could be used to justify the existence of exhaustive and non exhaustive injections), the existence of just the sets from which they have been deduced. Here, we are simply discussing if the method of pairing the elements of two sets is appropriate to compare their respective cardinality; and if it is, why non-exhaustive injections are rejected, because that rejection could be concealing a fundamental contradiction.

9 Assume, only for a moment, that exhaustive and non exhaustive injections were valid instruments to compare the cardinality of any two sets. In those conditions, let B be an infinite set. By definition, there exists a proper subset A of B and an exhaustive injection f from A to B proving both sets have the same number of elements. Consider now the injection g from A to B defined by:

$$g(x) = x, \forall x \in A \tag{1}$$

which evidently is non-exhaustive (the elements of the nonempty set $B-A$ remain unpaired). The injections f and g would be proving that A and B have (f) and not have (g) the same number of elements, i.e. that infinite sets are inconsistent.

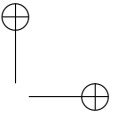
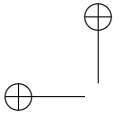
10 We must therefore decide if exhaustive and non-exhaustive injections do have the same validity as instruments to compare the number of elements of any two sets. If they have, then actually infinite sets are inconsistent. If they don't, at least one reason should be given to explain why they don't. And, if no reason can be given, then the arbitrary distinction in favor of the exhaustive injections should be arbitrarily declared in an appropriate ad hoc axiom. Until then, the foundation of (transfinite) set theory rests on the basis of one of the terms of a possible contradiction.⁶

11 As might be expected from a theory with such a foundation, in-

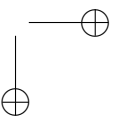
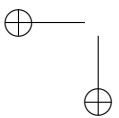
⁶Unbelievable as it may seem, the axiomatic foundation of set theory has always ignored this problem.

consistencies appeared from its very beginning: the set of all ordinals and the set of all cardinals were proved to be inconsistent by Burali-Forti [4] and Cantor respectively. According to Cantor those sets are inconsistent because of their excessive infinitude.⁷ One can be infinite but only within certain limits. By the appropriate ad hoc axioms, it was finally stated that some infinite totalities, as the totality of cardinals or the totality of ordinals, do not exist because they lead to contradictions. It can easily be proved, however, that in a naive (not limited by axiomatic restrictions) set theory, as Cantor’s set theory, each set of cardinal C originates nothing less than 2^C inconsistent totalities (see next chapter).

⁷Letter to Dedekind quoted in [5, pag. 245], [11], [9].



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