1.-Rationals from irrationals

n-Expofactorial numbers

1 This chapter introduces n-expofactorial numbers and the method of the successive decimal expansions by means of which it is possible to define a different rational number from the infinite decimal expansion of each irrational number within the real interval (0, 1). The implications of this unexpected result are briefly discussed.

2 Although the method of the successive decimal expansions we will make use of in the next section works with natural numbers of any size, we will use natural numbers unimaginably large: the n-expofactorials numbers defined in 4.

3 The expofactorial[1] of a natural number n, written $n^!$ (note the factorial symbol '!' is written as a superscript), is the factorial $n!$ raised $n!$ times to the power of $n!$ So, while the expofactorial of 2 is 16, the expofactorial of 3 is:

$$3^! = 6^{6^{6^{6^{6^{6}}}}} = 6^{6^{6^{6^{46656}}}} = 6^{6^{6^{26591197721532267796824894043879\ldots}}}$$

where the incomplete exponent of the last term on the right has noth-

[1]The first time I considered this type of numbers, I didn't know they have already been defined by C. A. Pickover ([2] cited in [4]) with the name of *superfactorials* and the symbols n$, the same name and symbols used by Sloane and Plouffe to define $n\$ = \prod_{k=1}^{n} k!$ [4]. So, I will retain my original notation $n^!$ and the name *expofactorial* for this type of numbers.

ing less than 36306 digits (roughly ten pages of standard text). The expofactorial of any natural number greater than 2 is so large that will probably never be calculated with exactitude (it is not an anodyne power of ten but a precise sequence of different digits).

4 Expofactorials are insignificant compared with n-expofactorials, recursively defined from expofactorials as follows: the 2-expofactorial of a natural number n, denoted by $n^{!2}$, is the expofactorial $n^{!}$ raised $n^{!}$ times to the power of $n^{!}$; the 3-expofactorial of n, denoted by $n^{!3}$, is the 2-expofactorial of n raised $n^{!2}$ times to the power of $n^{!2}$; the 4-expofactorial of n, denoted by $n^{!4}$, is the 3-expofactorial of n raised $n^{!3}$ times to the power of $n^{!3}$; and so on:

$$n^{!} = n!^{n!^{(\overset{n!}{\dots})^{n!}}} \qquad n^{!2} = {n^{!}}^{{n^{!}}^{(\overset{n^{!}}{\dots})^{n^{!}}}} \qquad n^{!3} = {n^{!2}}^{{n^{!2}}^{(\overset{n^{!2}}{\dots})^{n^{!2}}}} \qquad n^{!4} = {n^{!3}}^{{n^{!3}}^{(\overset{n^{!3}}{\dots})^{n^{!3}}}} \qquad \dots$$

The *grandeur* of, for example, $9^{!9}$ (9-expofactorial of 9) is far beyond human imagination. Three standard arithmetic symbols, just $9^{!9}$, is all we need to define a *finite* number so large that the standard writing of its precise sequence of digits would surely require a volume of paper much more greater than the volume of the visible universe.

5 The discussion that follows makes use of the 9-expofactorial of 9. For simplicity, it will be denoted by the letter 'k'. So, in what follows k will stand for $9^{\prime 9}$.

An irrational source of rational numbers

6 According to the hypothesis of the actual infinity, subsumed within the Axiom of Infinity, the infinite decimal expansion $0.d_1 d_2 d_3 \dots$ of any real number (with an infinite decimal expansion) within the real interval (0, 1) does exist as a complete $\omega-$ordered totality: it has a first digit, d_1, and each digit d_n (except d_1) has an *immediate predecessor*

d_{n-1} and an *immediate successor* d_{n+1}, so that no last digit exists. Since the argument that follows deals exclusively with $\omega-$ordered infinities, from now on, and for simplicity, they will be referred to simply as infinities.

7 A point of note is that ω, the ordinal of the $\omega-$ordered sequences, is *the less infinite ordinal*. Therefore, if r and s are two real numbers within the real interval $(0, 1)$ and they coincide in their first successive ω digits, then both numbers are identical. On the contrary, and taking into account that between any finite ordinal and ω only other finite ordinals do exist, if r and s are different then they can only coincide in a finite number of their first successive digits.

8 Let $\mathbb{N}$ be the set of natural numbers, k the 9-expofactorial of 9, and m_α any element of the set M of the irrational numbers within the real interval (0, 1). The exclusive decimal expansion:

$$m_\alpha = 0.d_1 d_2 d_3 \ldots \tag{1}$$

of m_α defines the following $\omega-$ordered sequence $\langle q_{\alpha,nk} \rangle_{n\in\mathbb{N}}$ of rational numbers:

$$\begin{aligned}
q_{\alpha,k} &= 0.d_1 d_2 \ldots d_k && (2)\\
q_{\alpha,2k} &= 0.d_1 d_2 \ldots d_k d_{k+1} \ldots d_{2k} && (3)\\
q_{\alpha,3k} &= 0.d_1 d_2 \ldots d_k d_{k+1} \ldots d_{2k} d_{2k+1} \ldots d_{3k} && (4)\\
&\ldots && (5)\\
q_{\alpha,nk} &= 0.d_1 d_2 \ldots d_k d_{k+1} \ldots d_{2k} d_{2k+1} \ldots d_{3k} d_{3k+1} \ldots d_{nk} && (6)\\
&\ldots && (7)
\end{aligned}$$

being $q_{\alpha,nk}$ (for every n in $\mathbb{N}$) the rational number within (0, 1) whose finite decimal expansion $0.d_1 d_2 \ldots d_{nk}$ coincides with the first nk digits of m_α. For this reason, m_α will be said the *source* of the sequence $\langle q_{\alpha,nk} \rangle_{n\in\mathbb{N}}$, and α will appear as a part of the subindex of each $q_{\alpha,nk}$. The rational $q_{\alpha,(n+1)k}$ will be said the k-expansion of the rational $q_{\alpha,nk}$ because $q_{\alpha,nk}$ is expanded with the next k successive digits (starting from d_{nk+1}) of the source m_α in order to define $q_{\alpha,(n+1)k}$. Don't forget the unimaginable grandeur of $k = 9^{!9}$.

9 From the perspective of the actual infinity hypothesis, the result of defining the infinitely many natural numbers by adding infinitely many successive times one unit to the first natural number 1, defines infinitely many increasing finite numbers, without ever reaching an infinite number.[2] Consequently, and being k a natural number, the result of defining the infinitely many elements of $\langle q_{\alpha,nk}\rangle_{n\in\mathbb{N}}$ by adding infinitely many successive times k new digits to the decimal expansion of $q_{\alpha,k}$, yield infinitely many finite decimal expansions (rational numbers), explosively increasing but always finite, without ever reaching an infinite decimal expansion.

10 This infinitist assumption will be essential for the next argument since it legitimates the actual existence of the *infinitely many* rational numbers in $\langle q_{\alpha,nk}\rangle_{n\in\mathbb{N}}$, all of them with *finitely many digits*, nk for each $q_{\alpha,nk}$. In the same way $\mathbb{N}$ contains infinitely many finite natural numbers, each one unit greater than its immediate predecessor, $\langle q_{\alpha,nk}\rangle_{n\in\mathbb{N}}$ contains infinitely many rational numbers with a finite decimal expansion, each with k digits more than its immediate predecessor. This is, in fact, infinitist orthodoxy.

11 Let P be the set of *all* pairs whose first component is a different element m_α of the set M of irrational numbers in (0, 1), and whose second component is the rational number $q_{\alpha,k}$ within (0, 1) defined by the first k successive digits $d_1, d_2, \dots d_k$ of m_α:

$$(m_\alpha,\ q_{\alpha,k}) \in P \Leftrightarrow \begin{cases} m_\alpha = 0.d_1 d_2 \dots d_k d_{k+1} \cdots \in M \\ \quad and \\ q_{\alpha,k} = 0.d_1 d_2 \dots d_k \end{cases} \tag{8}$$

Although the first element m_α of each pair is a different irrational number, the second one $q_{\alpha,k}$ will be repeated a certain number of times in the different pairs of P.

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[2]The recursive definition of natural numbers in set theoretical terms leads to the same conclusion

12 Notice that if there is no irrational number in $(0,1)$ with the same first k decimal digits, then the second element of each pair of P would be a different rational number. In these conditions the discussion that follows would be unnecessary: there would be as many rationals as irrationals within $(0,1)$.

13 Let now $q_{\alpha,k}$ be any of the repeated rationals in P, and let P_α be the subset (of P) of all pairs $(m_\varphi,\ q_{\varphi,k})$ whose second rational component $q_{\varphi,k}$ coincides with $q_{\alpha,k}$:

$$P_\alpha = \{(m_\varphi,\ q_{\varphi,k})\ |(m_\varphi,\ q_{\varphi,k}) \in P\ \wedge\ q_{\varphi,k} = q_{\alpha,k}\ \} \tag{9}$$

For simplicity, the repeated rational numbers in P_α will be called P-repetitions. Each pair $(m_\varphi, q_{\varphi,k})$ defines a sequence $\langle q_{\varphi,nk}\rangle_{n\in\mathbb{N}}$ of rational numbers similar to the sequence $\langle q_{\alpha,nk}\rangle_{n\in\mathbb{N}}$ defined in 8, except that the source is now the irrational number m_φ, so that $q_{\varphi,(n+1)k}$ is the k-expansion of its immediate predecessor $q_{\varphi,nk}$, for all n in $\mathbb{N}$.

14 By definition, the irrational numbers of all pairs of P_α are irrationals numbers within $(0,1)$ with the same first k digits. Obviously, some of these numbers will also have the first $2k$ digits and some will not.[3] Of these, some will have the first $3k$ digits and some will not. And so on and on. Recall that k is the 9-expofactorial of 9, a number inaccesible to human imagination.

15 The actual existence, all at once, of the infinitely many digits of the $\omega-$ordered decimal expansion of any irrational number in $(0,1)$ as a complete totality, legitimates the definitions of the sets P, P_α, as well as the sequences $\langle q_{\varphi,nk}\rangle_{n\in\mathbb{N}}$, all of them as complete totalities. It also legitimates the following:

Definition 15.-*A k-replacement of P_α is an operation on the set P_α which consists in the following two steps:*

1. *Each repeated rational in P_α is replaced with its k-expansion, what originates a new set P'_α.*

[3] Change, for instance, any digit $d_{(k+i)0<i\leq k}$ in any irrational in $(0,1)$ and you will get an irrational with the same first k digits but not with the same $2k$ digits.

2. *The set P_α is redefined as P'_α.*

16 As a consequence of 14, once k-replaced, not all rationals in P_α will be repeated. In those conditions, we can write:

$$P_\alpha = P^o_\alpha \cup P^*_\alpha; \quad P^o_\alpha \cap P^*_\alpha = \emptyset \tag{10}$$

where P^o_α is the subset of P_α whose pairs contain no repeated rational, and P^*_α is the subset of P_α whose pairs contain the same repeated rational.

17 Once k-replaced, P_α can be k-replaced again if $P^*_\alpha \neq \emptyset$, which means the repeated rational in each $(m_\varphi, q_{\varphi,2k})$ of P^*_α is replaced with its corresponding k-expansion $q_{\varphi,3k}$. And whenever the resulting P^*_α be not empty, a second, third, fourth etc. k-replacements can be carried out. Thus we can define the following sequence $\langle R_n \rangle_{n\in\mathbb{N}}$ of k-replacements:

$$n = 1, 2, 3, \dots \begin{cases} \text{If } P^*_\alpha \neq \emptyset \text{ then:} \\ 1: \ R_n(P_\alpha) = P^o_\alpha \cup \{(m_\varphi, q_{\varphi,(n+1)k}) | (m_\varphi, q_{\varphi,nk}) \in P^*_\alpha\} \\ 2: \ P_\alpha = P^o_\alpha \cup R_n(P_\alpha) \end{cases} \tag{11}$$

While $P^*_\alpha \neq \emptyset$, each $R_n(P_\alpha)$ is in fact well defined because for each $(m_\varphi, q_{\varphi,nk})$ in P^*_α and each n in $\mathbb{N}$, the $(n+1)$th terms $q_{\varphi,(n+1)k}$ of $\langle q_{\varphi,nk} \rangle_{n\in\mathbb{N}}$ does exist, making it possible to replace $q_{\varphi,nk}$ with it. Evidently, the successive k-replacements increase the number of non repeated rationals to the detriment of the repeated ones.

18 Let us assume now that while $P^*_\alpha \neq \emptyset$ and P_α can be k-replaced, it is k-replaced. Once all possible k-replacements have been carried out, there will be two mutually exclusive alternatives regarding P^*_α:

1. P^*_α is not empty.
2. P^*_α is empty.

Consider the first alternative: P^*_α is not empty. We know that for each element $(m_\lambda, q_{\lambda,vk})$ in P^*_α there is an ω−ordered sequence $\langle q_{\lambda,nk} \rangle_{n\in\mathbb{N}}$ of rationals with a finite decimal expansion. So that each $(m_\lambda, q_{\lambda,vk})$ in P^*_α can be replaced with $(m_\lambda, q_{\lambda,(v+1)k})$. Consequently a new k-replacement of P_α is possible, which contradict the fact that, being $P^*_\alpha \neq \emptyset$, all possible k-replacements of P_α have been carried out. Therefore, and by Modus Tollens, The first alternative is false and then, once performed all possible k-replacements of P_α the set P^*_α is empty.

Remarck 1 Note that argument 18 has nothing to do with constructive reasonings based on the successively performed k-replacements. It is a single Modus Tollens: once performed all possible k-replacements, the hypothesis that P^*_α is not empty leads to the contradictory conclusion that not all possible k-replacements have been carried out. That hypothesis must be, therefore, false.

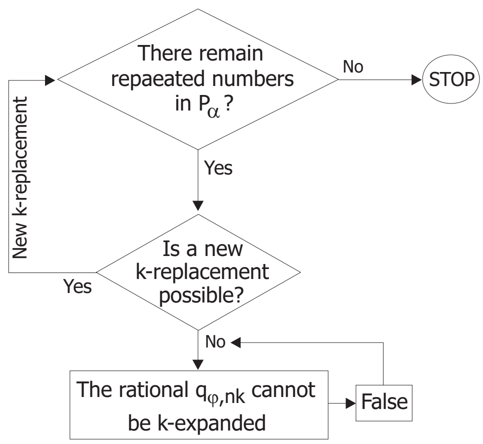

Figure 1.1: The consequences of being completed without a last completing element.

Remarck 2 Argument 18 takes advantage of the fact that, in accord with the hypothesis of the actual infinity, $\omega-$ordered sequences do exist as complete totalities despite the fat that no last element completes them (Figure 1.1). This assumption, makes it possible to ensure that while P_α contains P-repetitions, i.e. while P^*_α is not empty, the repeated numbers can be replaced with their corresponding successive k-expansions by means of successive k-replacements of P_α.

And that this sequence of k-replacements can *actually be completed* because of the *actual completeness* of each $\langle q_{\varphi,nk}\rangle_{n\in\mathbb{N}}$. Consequently, only when P_α no longer contains P-repetitions, i.e. when P^*_α is empty, will it be possible to ensure that all possible k-replacements have been carried out (under penalty of contradiction).

Remarck 3 In contrast, from the potential infinity perspective the existence of completed infinite totalities without a last element that completes them, makes no sense. Thus, from this perspective we are not legitimated to consider the completion of the sequence of k-replacements if this sequence is potentially infinite.

19 Once removed all P-repetitions, the resulting rational numbers can only have a finite decimal expansion since all elements of all sequences $\langle q_{\varphi,nk}\rangle_{n\in\mathbb{N}}$ are rational numbers with a finite expansion.

20 In accordance with the definition 13 of P_α, the rational numbers resulting from the removal of all P-repetitions cannot be repeated in the set $P - P_\alpha$ because all rational numbers in this last set differ from the rationals of P_α in at least one of their first k digits.

21 The above argument 13/20 can be applied to any other repeated rational in the set P of pairs $(m_\alpha, q_{\alpha,nk})$. In consequence, all repeated rationals can be replaced with a different rational number derived from the decimal expansion of the first irrational component of the pair. In these conditions each pair of P will be formed by a different irrational m_α and a different rational number q_α. The one to one correspondence f defined by:

$$f(m_\alpha) = q_\alpha \tag{12}$$

would be proving the set of rationals and the set of irrationals in $(0, 1)$ have both the same cardinality.

DISCUSSION

22 The hypothesis of the actual infinity subsumed by the Axiom of Infinity legitimates the following line of reasoning on which argument 11/21 is grounded:

1 The infinitely many digits of the decimal expansion of any irrational number within $(0, 1)$ do exist as an actual complete totality.

2 The infinite decimal expansions of the irrational numbers in $(0, 1)$ can only be $\omega-$ordered, being ω the less infinite ordinal.

3 Two different irrational numbers in $(0, 1)$ can only coincide in a finite number of their first successive decimal digits.

4 The infinitely many k-expansions $\langle q_{\varphi,nk} \rangle_{n \in \mathbb{N}}$ defined from the decimal expansion of each irrational m_φ in the real interval $(0, 1)$ do exist as an actual complete totality.

5 Each of the infinitely many k-expansions $\langle q_{\varphi,nk} \rangle_{n \in \mathbb{N}}$ is a rational number with finitely many digits.

6 In accordance with 4 and 5, the repeated rationals of P_α can be successively replaced with their corresponding successive rational k-expansions any finite or infinite number of times.

7 In these conditions, and by Modus Tollens 18, all P-repetitions can be removed from P_α, and then from P, so that each pair will be formed by a different irrational and a different rational derived from its irrational partner.

8 Consequently each irrational number within (0, 1) defines a different rational number within the same interval.

23 Conclusion 22-8 contradicts other well known results on the cardinality of the set of rational numbers [1].

24 To define rational numbers, and $\omega-$ordered sequences of rational numbers, from the decimal expansion of the irrational numbers leads to some other contradictory results we have not dealt with here.

EPILOG

25 As it has been repeatedly said, from the perspective of the actual infinity hypothesis, the infinitely many digits of a real number with an infinite decimal expansion do exist as a complete $\omega-$ordered totality. In consequence they can only be mind-independent entities (human mind cannot embrace the actual infinity). Thus, from the infinitist perspective, all real numbers would be (platonic) mind-independent entities.

26 From the hypothesis of the potential infinity, however, an irrational number is not a mind-independent entity formed by a complete $\omega-$ordered sequence of digits that exist all at once and by themselves. From this hypothesis, irrational numbers result from endless processes of calculus that cannot be replaced with a division between two integers, although at each stage of the calculus the number coincides with a rational number of finitely many digits. In this sense irrational numbers are also definable as (potentially infinite) sequences of rational numbers.

27 In the case of the rational numbers the processes of calculus can be replaced with a division between two integers, which is not necessarily endless. In its turn, integer numbers would result from the endless process of counting. Naturally, the existence of endless processes of calculus or of counting does not necessarily mean the existence of their corresponding finished results, as is assumed from the infinitist point of view.

28 We should decide which of the two alternatives is the most appropriate to found a theory of numbers. And the election is not an irrelevant one: we need mathematics to explain the world. Think, for example, of the problems posed by the actual infinity in certain areas of physics, as the standard model of particles (*renormalization*) or quantum gravity [3]. Or the assumed dense ordering of the *continuum* spacetime[4] versus the discontinuous nature of ordinary matter, electric charge or energy.

[4]Founded on the assumed uncountable cardinality $2^{\aleph_o}$ of the real numbers.

Bibliography

[1] Georg Cantor, *Über unendliche lineare Punktmannigfaltigkeiten*, Mathematischen Annalen **15** (1879), 1–7.

[2] Clifford A. Pickover, *Keys to Infinity*, Wiley, New York, 1995.

[3] Lee Smolin, *The trouble with physics*, Allen Lane. Penguin Books, London, 2007.

[4] Eric W. Weistein, *Superfactorial*, Eric Weisstein World of Mathematics, Wolfram Research Inc., http://mathworld.wolfram.com, 2009.