1.-Nested set inconsistency

A denumerable version of nested-set theorem

1 Let $A_1 = \{a_1, a_2, a_3 \dots \}$ be any $\omega-$ordered set and consider the following recursive definition:

$$A_{i+1} = A_i - \{a_i\}; \ i = 1, 2, 3, \dots \tag{1}$$

that yields the $\omega-$ordered sequence of nested sets:

$$S \ = \ A_1 \supset A_2 \supset A_3 \supset \dots \tag{2}$$

being each set $A_n = \{a_n, a_{n+1}, a_{n+2}, \dots\}$ a denumerable proper subset of all its predecessors, as well as a superset of all its successors.

2 The sequence S of sets $\langle A_n \rangle_{n \in \mathbb{N}}$ satisfies:

$$\bigcap_S A_i = \emptyset \tag{3}$$

where $\bigcap_S A_i$ stands for the intersection of all sets of the sequence S. This result is a *denumerable version* of the nested-sets theorem[1] that, for the sake of simplicity, will be referred to as NST. The proof of NST is immediate: if an element a_k would belong to the intersection then only a finite number (equal or less than k) of sets would have been

[1]The original version deals with non-denumerable sets, and the conclusion is exactly the contrary, i.e. that the intersection is nonempty.

defined by (1), since a_k does not belong to A_{k+1}, A_{k+2}, A_{k+3}, ...

3 NST is a trivial result in modern infinitist mathematics. As far as I know, it has never been involved in any discussion on the formal nature of infinity. The theorem simply states the sets $\langle A_n \rangle_{n \in \mathbb{N}}$ have no common element. The implications of the fact that *each* A_i is a denumerable proper subset of *all* its predecessors have never been examined. In the next discussion we will have the opportunity to examine some of those implications.

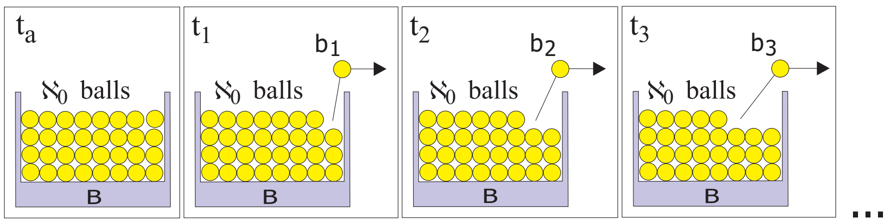

Figure 1.1: Removing, one by one, the balls of a box that contains $\aleph_o$ balls.

4 Before beginning our discussion, let us examine an elementary *physical* version of NST. Let B be a box containing a countable collection of balls labeled as b_1, b_2, b_3, . . . , and let $\langle t_n \rangle_{n \in \mathbb{N}}$ be a strictly increasing ω−ordered sequence of instants within the real interval (t_a, t_b) whose limit is just t_b. Now consider the following supertask: at each instant t_i remove the ball b_i, and only it, from the box. The one to one correspondence f between $\langle t_n \rangle_{n \in \mathbb{N}}$ and $\langle b_n \rangle_{n \in \mathbb{N}}$ defined by $f(t_i) = b_i$ proves that at t_b all balls will have been removed from B.

5 In accordance with the way of removing the balls, one by one and in such a way that between the removal of a ball and the removal of the next one an interval of time greater than zero always elapses, it could be expected that just before completing the removal of all balls from the box, the box will contain . . . 5, 4, 3, 2, 1, 0 balls. Nothing further from the (infinitist) truth: the box will never contain a finite number n of balls, whatever be n, simply because these balls would be the impossible n last balls of an ω−ordered collection of balls, and the successive instants at which they would be removed would be the impossible last n instants of an $\omega-$ordered sequence of instants.

6 Let $f(t)$ be the number of balls within the box at any instant t in $[t_a, t_b]$, i.e. the number of balls to be removed at the precise instant t. As a consequence of $\omega-$order, we will have the following inevitable dichotomy:

$$f(t) = \begin{cases} \aleph_o, \ \forall t \in [t_a, t_b) \\ 0 \text{ if t} = t_b \end{cases} \tag{4}$$

Otherwise, if for a t in $[t_a, t_b)$ we would have $f(t) = n$, being n any natural number, then there would exist the impossible last n terms of an $\omega-$ordered sequence.

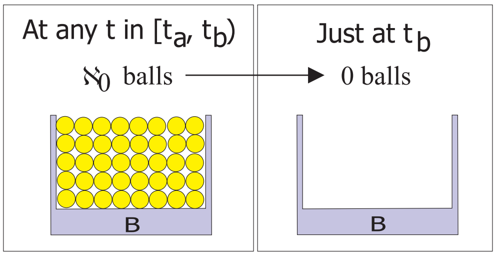

Figure 1.2: The 'Aleph-zero' or zero dichotomy

7 Taking into account the one to one correspondence $f(t_i) = b_i$, all balls $\langle b_n \rangle_{n \in \mathbb{N}}$ are removed *one by one* from the box B, one after the other and in in such a way that an interval of time $\Delta_i t = t_{i+1} - t_i$ greater than zero always elapses between the removal of two successive balls b_i, b_{i+1}, $\forall i \in \mathbb{N}$. But according to the above $\aleph_o$ or 0 dichotomy (4), this is impossible because the number of balls to be removed from the box has to change *directly*[2] from $\aleph_o$ to 0, and this is only possible by removing simultaneously $\aleph_o$ balls.

8 Evidently, the box B plays the role of the set A_1 and the successive removal of balls represent the successive steps of the recursive defi-

[2]Without intermediate finite states at which only a finite number of balls remain to be removed.

nition $A_{i+1} = A_i - \{a_i\}$. Since the successive elements a_1, a_2, a_3, ... of A_1 are successively removed in order to define the successive terms A_1, A_2, A_3, ... of the sequence S, we could write:

$$\{\not{a}_1, a_2, a_3, a_4, \ldots\} \tag{5}$$

$$\{\not{a}_1, \not{a}_2, a_3, a_4, \ldots\} \tag{6}$$

$$\{\not{a}_1, \not{a}_3, \not{a}_3, a_4, \ldots\} \tag{7}$$

$$\ldots \tag{8}$$

where $\not{a}_1, \not{a}_2, \not{a}_3, \ldots$ simply indicate the successive elements a_1, a_2, a_3, ... of A_1 that have been used in order to define the successive members A_2, A_3, A_4, ... of the sequence S.

9 As in the case of the box B, and for the same reasons, if we focus our attention on the number of elements that remain unmarked in (5)-(8) as the recursive definition (1) progresses, then we will immediately come to the conclusion that that number can only take two values: $\aleph_o$ and 0.

10 The $\aleph_o$ or 0 dichotomy implies the number of unmarked elements in (5)-(8) changes directly from $\aleph_o$ to 0, and this is only possible by marking $\aleph_o$ elements at once, i.e. by defining simultaneously $\aleph_o$ sets of the sequence S, which evidently is not compatible with the recursiveness of that definition.

11 There is, however, a significant difference between taking away the balls from B and recursive definition (1): while the box B is always the same as the balls are successively removed from it (which makes it evident the fallacy of the removal), the set A_1 originates a sequence of sets: starting from A_1, each set A_i originates a new set A_{i+1} when the element a_i is removed from it in order to define the next term of the sequence. Thus, A_1 dissolves in a complete infinite sequence of sets without a last set completing the sequence, which hides the fallacy of removing one by one all elements of a collection without ever resting ... three, two, one, elements to be removed.

NESTED-SET INCONSISTENCY

12 The above discussion on NST suggests this theorem is not as trivial as it seems to be. It, in fact, motivates the short discussion that follows, whose main objective is to put into question the formal consistency of the the actual infinity hypothesis.

13 It seems convenient at this point to recall that Cantor took it for granted the existence of the set of all finite cardinals as a complete infinite totality (Axiom of Infinity in modern terms), and that from that initial assumption he successfully derived the infinite sequence of the transfinite ordinals of countable well-ordered sets built on ω, the smallest of them [1, Theorem 15-K]. Thus any result affecting the formal consistency of ω will affect the whole sequence of transfinite ordinals of the second class as well as the formal consistency of the actual infinity hypothesis subsumed within the Axiom of Infinity.

14 Let us just begin by assuming the Axiom of Infinity and then the existence of $\omega-$ordered sets and $\omega-$ordered sequences as complete infinite totalities.

15 Consider again the above sequence of sets $S = A_1, A_2, A_3, \ldots$. From S we will define the sequence S^* of sets by:

$$n = 1, 2, 3, \ldots \begin{cases} n = 1 : S^* = A_1 \\ n > 1 : A_1 \cap A_2 \cap \cdots \cap A_n \neq \emptyset \Rightarrow S^* = A_1, A_2, \ldots\ A_n \end{cases} \tag{9}$$

16 As in previous arguments in this book, it could easily be proved by induction or by Modus Tollens that for any natural number v the first v definitions (9) can be carried out.

17 Assume that while the successive definitions (9) can be carried out they are carried out. Once all possible definitions (9) have been carried out, the sequence S^* will be formed by a certain number (finite or infinite) of sets that by definition have a nonempty intersection. Let, therefore, a_v be any element of that intersection. Evidently, we have:

$$a_v \notin A_{v+1} \tag{10}$$

And in consequence A_{v+1} is not a member of the sequence S^*.

18 But, on the other hand, we also have:

$$A_1 \cap A_2 = A_2 \neq \emptyset \tag{11}$$

$$A_1 \cap A_2 \cap A_3 = A_3 \neq \emptyset \tag{12}$$

$$\ldots \tag{13}$$

$$A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_v \cap A_{v+1} = A_{v+1} \neq \emptyset \tag{14}$$

Thus, in accordance with 16, the first v definitions (9) have been carried out. Consequently A_{v+1} is in S^*

19 We have, therefore, derived a contradiction from our initial assumption: the set A_v is and is not in the sequence S^*.

20 The alternative to the above contradiction is another contradiction even more elemental: after having performed all possible definitions (9), not all possible definitions (9) have been performed.

Bibliography

[1] Georg Cantor, *Contributions to the founding of the theory of transfinite numbers*, Dover, New York, 1955.