1.-Jordan curves of infinite length

INTRODUCTION

1 The ω−ordered sequence $\langle z_n \rangle_{n\in\mathbb{N}}$ of Z-points within the real interval $(0, 1)$ defined by:

$$z_n = \frac{2^n - 1}{2^n} \tag{1}$$

is an example of ω-partition of a finite line segment. Each pair of successive points x_n, x_{n+1} defines a part of the partition. The successive parts are adjacent and so that the right end of any one of them coincides with the left end of the following one:

$$[x_1, x_2),\ [x_2, x_3),\ [x_3, x_4),\ \ldots \tag{2}$$

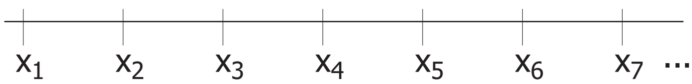

Figure 1.1: Partition of a straight line.

2 As is well known, at least since the 18th century, ω-partitions of finite line segments are only possible if the successive adjacent parts of the ω-partition are of a decreasing length, otherwise the length of the line would have to be infinite [1]. This inevitable restriction originates a huge asymmetry in the partition. Indeed, whatever be the length of the ω-partitioned segment AB and whatever be the ω-

partition, all its parts, except a finite number of them, will necessarily lie within a final interval *CB* arbitrarily small .

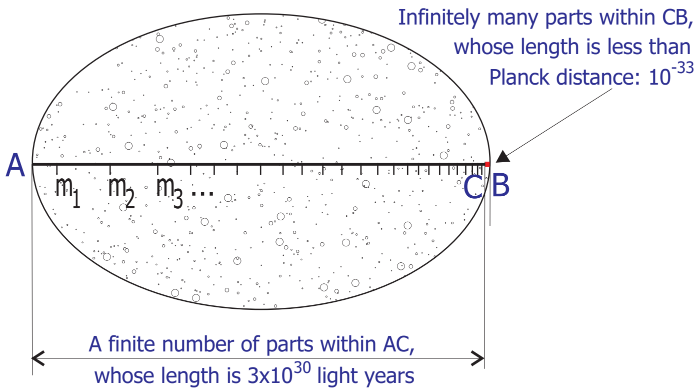

Figure 1.2: Spacial ω-asymmetry in the ω -partition of a line segment *AB* whose length is the diameter of the visible universe.

3 For the sake of illustration, consider an ω-partition of a line segment *AB* whose length is 10^{30} light years, the assumed diameter of the visible universe. Whatever be the ω-partition of this enormous segment all its infinitely many parts, except a finite number of them, will inevitably lie within a final interval *CB* inconceivable less than, for instance, Planck length ($\sim 10^{-33}$ cm). There is no way of performing a less unbiased partition if the partition has to be $\omega-$ordered, the smaller of the infinite partition (Figure 1.2). Thus, ω-partitions are ω-asymmetrical. And being ω the less infinite ordinal, any transfinite partition has to contain at least an $\omega-$ordered partition.

4 The unaesthetic consequence of the above asymmetry becomes a little more controversial if the partitioned object is a closed line as a Jordan curve. The objective of the following short discussion is just to examine such a partition.

Transfinite partition of a Jordan curve

5 Let $f(x)$ be a real valued function whose graph is a Jordan Curve[1] **J** in the euclidian plane $\mathbb{R}^2$. Let a and b be the endpoints of the arc $\widetilde{ab}$ in **J**. We will write $L(a,b)$ to denote the length of $\widetilde{ab}$:

$$L(a,b) = \int_a^b \sqrt{1 + (f(x)')^2}dx \tag{3}$$

6 Assume **J** has an infinite length. In these conditions let r be any proper real number greater than 0 and assume **J** is divided clockwise from any point x_1 into a certain number of adjacent parts $\widetilde{x_1x_2}$, $\widetilde{x_2x_3}$, $\widetilde{x_3x_4}$ ... so that each part $\widetilde{x_ix_{i+1}}$ has a *finite* length equal or greater than r:

$$L(x_i, x_{i+1}) \geq r, \ \forall i \in I \tag{4}$$

where I is the partition's set of indexes.

7 Evidently, the partition $\langle x_i \rangle_{i \in I}$ has to be infinite otherwise, and being finite the length of the parts, **J** would have a finite length. In addition, and according to Cantor [2], $\langle x_i \rangle_{i \in I}$ cannot be uncountably infinite. In fact, consider the sequence of real numbers $\langle r_i \rangle_{i \in I}$ defined as:

$$r_1 = x_1 \tag{5}$$

$$r_{i+1} = r_i + L(x_i, x_{i+1}), \ \forall i \in I \tag{6}$$

The one to one correspondence f between $\langle x_i \rangle_{i \in I}$ and $\langle r_i \rangle_{i \in I}$ defined by $f(x_i) = r_i$ proves that both sequences have the same cardinality. Thus, if the first one were uncountably infinite so would be the second. But $\langle r_i \rangle_{i \in I}$ cannot be uncountably infinite because if that were the case we could pick up a different rational number q_i in each real interval $[r_i, r_{i+1})$ and then we would have an uncountable set of different rational numbers, which, according to Cantor, is impossible.

8 Therefore, the partition $\langle x_i \rangle_{i \in I}$ has to be countably infinite. Accord-

[1] A Jordan curve is a simple closed line that is topologically equivalent to the unit circle, i.e. one that does not intersect itself.

ingly, the sequence $\langle x_i \rangle_{i \in I}$ can only be β-ordered, being β a transfinite numerable ordinal, i.e. an ordinal of the second class (first or second kind) according to Cantor's terminology.[2]

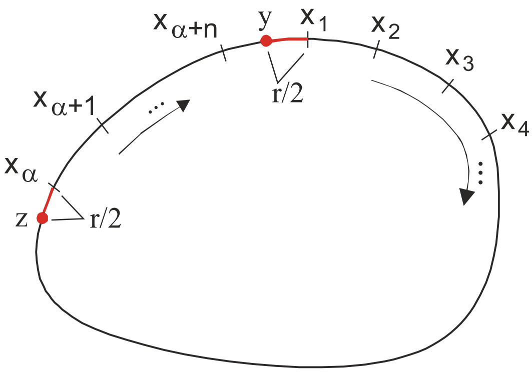

Figure 1.3: Transfinite partition of a Jordan Curve in the euclidian $\mathbb{R}^2$.

9 Now consider a point y anticlockwise from x_1 and such that:

$$L(y, x_1) = r/2 \tag{7}$$

From (4) we infer that y can only belong to the last part of $\langle x_i \rangle_{i \in I}$. So, this sequence must have a last part and then it must holds: $\beta = \alpha + n$, being α a transfinite ordinal of the second class second kind,[3] and n a finite ordinal.

10 In compliance with 9, $\langle x_i \rangle_{i \in I}$ can only be (α+n)-ordered. Consequently, an αth ordinal of the second class second kind has to exist in the set of indexes I. Therefore, and being α of the second kind, $\widetilde{x_\alpha x}_{\alpha+1}$ has not immediate predecessor in the partition.[4] Now then, according again to (4), the point z anticlockwise from x_α and such

[2]See Chapter **??** on the actual infinity.

[3]Ordinals of the second class first kind have immediate predecessor and immediate successor (as in $\gamma + 1$, $\gamma + 2$, $\gamma + 3$, etc.), while those of the second kind have immediate successor but not immediate predecessor because they are the limit of a sequence of ordinals of the first kind.

[4]α is the limit of a sequence of first kind ordinals, and being the limit of a

that $L(z, x_\alpha) = r/2$ can only belong to a part immediately preceding $\widetilde{x_\alpha x_{\alpha+1}}$, which is impossible. This proves that $\langle x_i \rangle_{i \in I}$ has to be, but cannot be, $(\alpha + n)$-ordered. A contradiction we have derived from our initial assumption on the infinite length of **J**. We therefore conclude that Jordan Curves of infinite length are inconsistent objects, and so will be any metrical space compatible with them.

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sequence it is not a term of the sequence, and then cannot have immediate predecessor.

Bibliography

[1] George Bekeley, *A Treatise Concerning the Principles of Human Knowledge*, Renascence Editions, http://darkwing.uoregon.edu/ bear/berkeley, 2004.

[2] Georg Cantor, *Über verschiedene Theoreme asu der Theorie der Punktmengen in einem n-fach ausgedehnten stetigen Raume Gn*, Acta Mathematica **7** (1885), 105–124.