

1.-Hilbert's machine

DEFINITIONS

1 Hilbert machine is a supertask machine inspired by Hilbert's Hotel whose functioning leads to a contradictory result involving ω -ordering and then the actual infinity hypothesis.

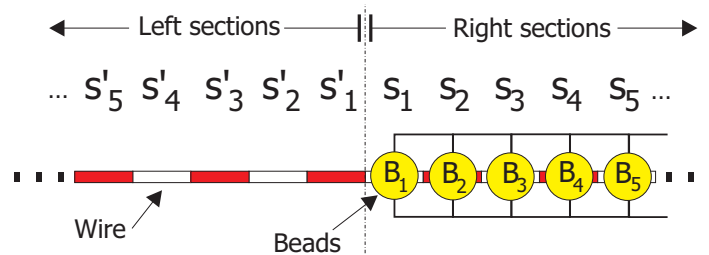


Figure 1.1: Hilbert's machine just before performing the first L-sliding.

2 In the following conceptual discussion we will make use of a theoretical device, inspired by the emblematic Hilbert Hotel, that will be referred to as *Hilbert machine*, composed of the following elements (see Figure 1.1):

1. An infinite horizontal wire divided into two infinite parts, the left and the right side:
 - (a) The right side in turn is divided into an ω -ordered sequence of adjacent sections $\langle S_n \rangle_{n \in \mathbb{N}}$ which are indexed

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from left to right as S_1, S_2, S_3, \dots . They will be referred to as right sections.

(b) The left side is also divided into an ω -ordered sequence of adjacent sections $\langle S'_n \rangle_{n \in \mathbb{N}}$ indexed now from right to left as \dots, S'_3, S'_2, S'_1 ; being S'_1 adjacent to S_1 . They will be referred to as left sections.

2. An ω -ordered sequence of beads $\langle b_n \rangle_{n \in \mathbb{N}}$ strung on the wire, which can slide on it as the beads of an abacus, being each bead b_i initially placed on the right section S_i .
3. A mechanism that slides simultaneously each bead exactly one section to the left. These simultaneous slidings one section towards the left will be referred to as L-slidings.
4. The first bead b_1 is equipped with a sensor E to detect empty left sections. L-slidings will be performed if, and only if, this sensor detects the left section adjacent to b_1 is empty.

3 Before performing an L-sliding, the sensor E detects if the left section adjacent to b_1 is empty, so that b_1 can slide to it and each bead $b_{i,i>1}$ to the section previously occupied by b_{i-1} . This way of functioning allows us to impose the following:

Restriction 3.-An L-sliding will be carried out if, and only if, all beads remain strung on the wire.

4 Since the segments $\langle S'_i \rangle_{i \in \mathbb{N}}$ of the left side of the wire are ω -ordered each segment S'_n has an immediate successor S'_{n+1} just on its left. In accord with the hypothesis of the actual infinity all those infinitely many left sections exist as a complete totality in spite of the fact that there is no last section completing the sequence.

5 Let us begin by proving that for each natural number v the first v L-slidings can be carried out. It is quite clear the first L-sliding can be carried out: the sensor E detects the empty left section S'_1 so that b_1 can slides to S'_1 and each $b_{i,i>1}$ to S_{i-1} . Assume that, being n any natural number, the first n L-slidings can be carried out. As a

consequence of those first n L-slidings, and taking into account that each L-sliding moves each bead just one section to the left, b_1 will be placed on S'_n . In these conditions E detects the empty left section S'_{n+1} so that b_1 can slide to S'_{n+1} and each bead $b_{i,i>1}$ to the section previously occupied by b_{i-1} . Consequently the $(n + 1)$ -th L-sliding can also be carried out. We have then proved the first L-sliding can be carried out, and if for any natural number n the first n L-slidings can be carried out, then the first $(n + 1)$ L-slidings can also be carried out. This finally proves that for each natural number v the first v L-slidings can be carried out.

HILBERT MACHINE CONTRADICTION

6 According to restriction 3, the successive L-slidings can be performed if, and only if, all beads remain strung on the wire. Assume then that while the successive L-slidings can be carried out, they are carried out. In these conditions we will prove the two following contradictory results.

7 The first result is the following:

Theorem 7.-Once performed all possible L-slidings at least one bead remains strung on the wire.

*Proof.-*Assume that once performed all possible L-slidings no bead remains strung on the wire. This would imply:

1. Infinitely many violations of restriction 3.
2. The existence of a last left section through which a bead can be removed from the wire by an L-sliding.

And both consequences are impossible: the second because a last left section would be needed in order to remove the beads from the wire, whereas the left sections $\langle S'_n \rangle_{n \in \mathbb{N}}$ of the wire are ω -ordered and then there is not a last left section. The first because arbitrary violations are unacceptable in formal theories. In this case it could be argued the infinitely many violations of Restriction 3 is an unexpected

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consequence of performing infinitely many L-slidings. But the same unexpected consequences could be expected from any other definition or procedure involving infinitely many steps, which, obviously, would invalidate the totality of transfinite mathematics.

8 The second result is the following:

Theorem 8. *Once performed all possible L-slidings no beads remains strung on the wire.*

Proof.—Let b_v be any bead and assume that once performed all possible L-slidings it remains in the right section S_k . It must be $k < v$ because all L-slidings are towards the left, the direction towards which the indexes of $\langle S_n \rangle_{n \in \mathbb{N}}$ decreases. Since b_v was initially on S_v only a finite number $v - k$ of L-slidings would have been performed, and then it would not have been possible to perform the the first $v - k + 1$ L-slidings, which goes against 5. A similar reasoning can be applied if b_v were finally placed on a left section S'_n , being now the number of performed L-slidings exactly $v+n-1$ and then it would not have been possible to perform the first $v+n$ L-slidings, which also goes against 5. Thus, if all possible L-slidings are performed no bead remain strung on the wire.

DISCUSSION

9 Let us compare the functioning of the above Hilbert machine (H_ω from now on) with the functioning of a finite version of the machine (symbolically H_n). This finite machine has a finite number n of both right and left sections (Figure 1.2). As in the case of H_ω , a finite sequence of n beads are initially placed in the right side of the wire, each bead b_i strung on the right section S_i . It is immediate to prove that H_n can only perform n L-slidings because not having a left section S'_{n+1} , the sensor E will stop the machine after the first n L-slidings. H_n halts with each left section S'_i occupied by the bead b_{n-i+1} and all right sections empty, and this is all. No contradiction is derived from the functioning of H_n . Thus for any natural number n , the cor-

responding machine H_n is a consistent theoretical artifact. Only the infinite Hilbert's machine H_ω is inconsistent.

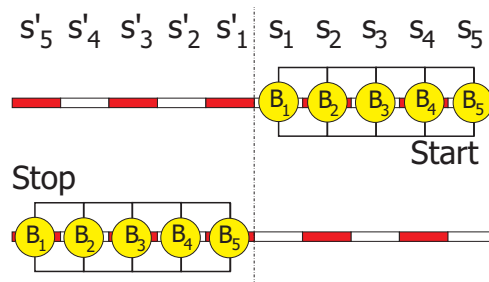
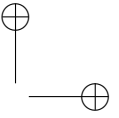
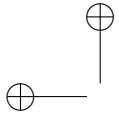
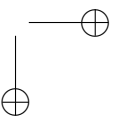
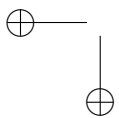


Figure 1.2: A finite machine of five sections.

10 What contradiction 7/8 proves is not the inconsistent functioning of a supermachine. What it proves is the inconsistency of ω -order itself. Perhaps we should not be surprised by this conclusion. After all, an ω -ordered sequence is one which is both complete (as the actual infinity requires) and uncompletable (there is not a last element that completes it). On the other hand, and as Cantor proved [1], [2], ω -order is an inevitable consequence of assuming the existence of infinite sets as complete totalities. An existence axiomatically stated in our days by the Axiom of Infinity, in all axiomatic set theories including ZFC and BNG [4], [3]. It is therefore that axiom the ultimate cause of contradiction 7/8.



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