1.-Extending Cantor paradox

INTRODUCTION

1 Cantor's paradox is really an inconsistency related to the set of all cardinals, once assumed such a set exists as a complete totality. For this reason, that set is explicitly rejected in modern axiomatic set theories. The following discussion proves, however, that not only the set of all cardinals is inconsistent, it proves that in Cantor's naive set theory each set of cardinality C originates at least 2^C inconsistent sets.

2 Although Burali-Forti was the first to publish an inconsistency related to transfinite sets [1], [6], Cantor was the first to discover a paradox in the nascent set theory: the maximum cardinal paradox [6], [4]. There is no agreement regarding the date Cantor discovered his paradox [6] (the proposed dates range from 1883 [9] to 1896 [7]). Burali-Forti paradox on the set of all ordinals and Cantor paradox on the set of all cardinals are both related to the size of the considered totalities, perhaps too big as to be consistent according to Cantor. It seems somewhat ironic that an infinite set may be inconsistent just because of its excessive size.

3 The simplest explanation for both paradoxes is that they really are inconsistencies derived from the hypothesis of the actual infinity, i.e. from assuming the existence of complete infinite totalities. But nobody has dared to analyze this alternative. It was finally accepted that some infinite totalities (as the set of real numbers) do exist while others (as the totality of cardinals, or the totality of ordinals) do not because they lead to contradictions.

Cantor paradox

4 Let us consider the following easy version of Cantor paradox[1]: Let U be the set of all sets, the so called universal set[2] and $P(U)$ its power set, the set of all its subsets. Let us denote by $|U|$ and $|P(U)|$ their respective cardinals. Being U the set of *all* sets, we can write:

$$|U| \geq |P(U)| \tag{1}$$

On the other hand, and according to Cantor theorem on the power set, [3] it holds:

$$|U| < |P(U)| \tag{2}$$

which contradicts (1). This is Cantor's inconsistency or paradox.

5 As is well known, Cantor gave no importance to that inconsistency [5] and clinched the argument by assuming the existence of two types of infinite totalities, the consistent and the inconsistent ones [2]. As noted above, in Cantor's opinion the inconsistency of those infinite totalities would surely due to their excessive size. We would be in the face of the mother of all infinities, the absolute infinity which leads directly to God, being just the divine nature of this absolute infinitude what makes it inconsistent for our poor human minds [2].

6 As we will immediately see, it is possible to extend Cantor paradox to other sets much more modest than the set of all sets. But neither Cantor nor his successors considered such a possibility. We will do it here. This is just the objective of the discussion that follows. A discussion that will take place within the framework of Cantor naive (non axiomatized) set theory.

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[1]Usual as it may be, the expression 'Cantor's paradox' is at least confusing since it is not a paradox but a true inconsistency.

[2]Cantor's naive set theory admits sets as the universal set U that are forbidden in modern axiomatic theories.

AN EXTENSION OF CANTOR'S PARADOX

7 Since the elements of a set in a naive set theory can be sets, sets of sets, sets of sets of sets and so on, we will begin by defining the following binary relation $\mathcal{R}$ between two sets: we will say that two sets A and B are $\mathcal{R}$-related, written $A \mathcal{R} B$, if B contains at least one element which forms part of the definition of at least one element of A. For instance, if:

$$A = \{ \{\{a, \{b\}\}\}, \{c\}, d, \{\{\{\{e\}\}\}\}, f\} \tag{3}$$

$$B = \{1, 2, b\} \tag{4}$$

$$C = \{1, 2, 3\} \tag{5}$$

then A is $\mathcal{R}$-related to B because the element b of B forms part of the definition of the element $\{\{ a, \{b\} \}\}$ of A, while A is not R-related to C because no element of C is involved in the definition of A's elements.

8 In these conditions, let X be any non empty set and Y any of its subsets. From Y we define the set $T_{\overline{Y}}$ according to:

$$T_{\overline{Y}} = \{Z \mid Z \cap Y = \emptyset \ \wedge \neg\exists V(V \cap Y \neq \emptyset \wedge Z \mathcal{R} V)\} \tag{6}$$

$T_{\overline{Y}}$ is, therefore, the set of all sets Z which contain no element of the set Y nor are $\mathcal{R}$-related to any set V that contains elements of the set Y. The argument that follows could also be applied to other less restrictive definitions of $T_{\overline{Y}}$.

9 Let us now consider the set $P(T_{\overline{Y}})$, the power set of $T_{\overline{Y}}$. The elements of $P(T_{\overline{Y}})$ are all of them subsets of $T_{\overline{Y}}$ and therefore sets of sets that neither contain elements of the set Y nor are $\mathcal{R}$-related to sets that contain elements of the set Y:

$$\forall D \in P(T_{\overline{Y}}): \ D \cap Y = \emptyset \wedge \neg\exists V(V \cap Y \neq \emptyset \wedge D \mathcal{R} V) \tag{7}$$

Consequently, it holds:

$$\forall D \in P(T_{\overline{Y}}): \ D \in T_{\overline{Y}} \tag{8}$$

And then:

$$P(T_{\overline{Y}}) \subset T_{\overline{Y}} \tag{9}$$

Accordingly, we can write:

$$|P(T_{\overline{Y}})| \leq |T_{\overline{Y}}| \tag{10}$$

10 On the other hand, and in accordance with Cantor's theorem it holds:

$$|P(T_{\overline{Y}})| > |T_{\overline{Y}}| \tag{11}$$

Again a contradiction. But now X is any non empty set and Y any of its subsets. We can therefore state that every set of cardinal C gives rise to at least 2^C inconsistent totalities, which is the number of subsets of a set with C elements.

11 The above argument not only proves the number of inconsistent infinite totalities is much greater than the number of consistent ones, it also suggests the excessive size of the sets could not be the cause of the inconsistency. Consider, for example, the set X of all sets whose elements are exclusively defined by means of the natural number 1:

$$X = \{1, \{1\}, \ \{1, \{1\}, \{1, \{1\}\}\}, \ \{\{\{1\}\}\}, \ \{\{ 1, \{1\} \}\} \dots \} \tag{12}$$

An argument similar to 8/10 would immediately prove it is an inconsistent totality, although compared to the universal set it is an insignificant totality.[3] However, it is also inconsistent.

12 Had we know the existence of so many inconsistent infinite totalities, and not necessarily so greater as the absolute infinity, and

[3] Recall, for instance, that between any to real numbers an uncountable infinity ($2^{\aleph_0}$) of other different reals numbers do exist. What, as Wittgenstein would surely say, makes one feel dizzy [11]

perhaps Cantor transfinite set theory would have been received in a different way. Perhaps the very notion of the actual infinity would have been put into question in set theoretical terms; and perhaps we would have discovered the way to prove it is an inconsistent notion. But, as we know, this was not the case.

13 The history of the set theory reception and the way of dealing with its inconsistencies (all of them promoted by the actual infinity hypothesis and by self-reference) is well known. From the beginnings of the XX century a great deal of effort has been carried out to found set theory on a consistent background free of inconsistencies. Although the objective could only be reached with the aid of the appropriate axiomatic patching. At least half a dozen of axiomatic set theories have been developed ever since.[4] Some hundred pages are needed to explain all axioms of contemporary axiomatic set theories. Just the contrary one could expect from the axiomatic foundations of a formal science.

14 As noted above, the simplest explanation of Cantor and Burali-Forti paradoxes is that they are true contradictions derived from the inconsistency of the hypothesis of the actual infinity. The same applies to the set of all sets, and to the set of all sets that are not member of themselves (Russell paradox), although in this case there is an additional cause of inconsistency related to the very definition of set. All sets involved in the paradoxes of naive set theory were finally removed from the theory by the opportune axiomatic restrictions. Nobody dared to suggest the possibility that those paradoxes were in fact contradictions derived from the hypothesis of the actual infinity; i.e. from assuming the existence of infinite sets as complete totalities.

15 What is really true is that Cantor set of *all* cardinals, Burali-Forti set of *all* ordinals, the set of *all* sets, and Russell set of *all* sets that are not members of themselves, are all of them inconsistent totalities when considered from the perspective of the actual infinity hypothe-

[4] There are also some contemporary attempts to recover naive set theory [8]

sis. Even Turing's famous halting problem is related to the hypothesis of the actual infinity because it also assumes the existence of all pairs (programs, inputs) as a complete infinite totality [10]. Under the hypothesis of the potential infinity, on the other hand, none of those totalities makes sense because from this perspective only finite totalities can be considered, indefinitely extensible, but always finite.

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