1.-Dichotomies

INTRODUCTORY DEFINITIONS

1 This chapter introduces a formalized version of each of Zeno's dichotomies based on the successiveness of $\omega-$order and ω^*-order. Each of these formalized versions leads to a contradiction.

2 In the second half of the XX century, several solutions to some of Zeno's paradoxes were proposed with the aid of Cantor's transfinite arithmetic, topology, measure theory and more recently internal set theory[1] (a branch of nonstandard analysis). It is also worth noting the solutions proposed by P. Lynds[2] within classical and quantum mechanics frameworks. Some of these solutions, however, have been contested. And in most of the cases, the proposed solutions do not explain where Zeno's arguments fail. Moreover, some of the proposed solutions gave rise to a new collection of problems so exciting as Zeno's paradoxes.[3]. In the discussion that follows I propose a new way of discussing Zeno's Dichotomies based on the notion of $\omega-$order, the order induced by ω, the first transfinite ordinal.

3 As is well known, an $\omega-$ordered sequence is one in which there is a first element and each element has an immediate successor and an immediate predecessor, except the first one. According to the hypothesis of the actual infinity subsumed within the Axiom of Infinity,

[1] [7], [8], [25], [9], [11], [10], [17], [16]
[2] [13], [14]
[3] [18], [1], [19], [20], [12] [21]

an ω−ordered sequence is a complete totality despite the fact that no last element completes it[4]. The sequence of natural numbers in their natural order of precedence is an example of ω−ordered sequence.

4 An ω^*−ordered sequence is one in which there exists a last element and each element has an immediate predecessor and an immediate successor, except the last one. From the same infinitist perspective, ω^*−ordered sequences are complete totalities in spite of the fact that there is not a first element to begin with. The increasing sequence of negative integers, . . . , -3, -2, -1, is an example of ω^*−ordered sequence.

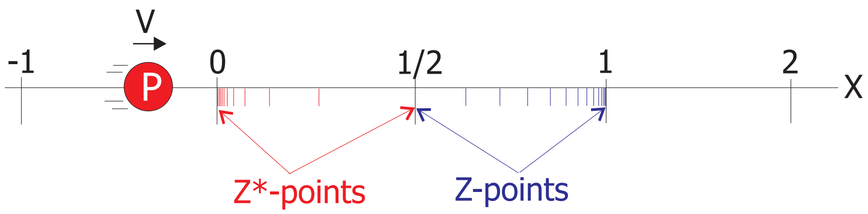

Figure 1.1: Z-points and Z^*-points.

5 Consider now a particle P moving through the X axis from the point -1 to the point 2 at a constant finite velocity v (Figure 1.1). Assume P is at point 0 just at the precise instant t_0. At instant $t_1 = t_0 + 1/v$ it will be exactly at point 1. Consider now the following ω−ordered sequence of Z-points [22] within the real interval $(0, 1)$, defined by

$$z_n = \frac{2^n - 1}{2^n}, \ \forall n \in \mathbb{N} \tag{1}$$

and the ω^*−ordered sequence of Z*-points within the same interval defined by:

$$z^*_{n*} = \frac{1}{2^n}, \ \forall n \in \mathbb{N} \tag{2}$$

[4] Cantor, in fact, proved the existence of ω−ordered sequences by assuming the existence of the set of all finite cardinals as a complete totality [5, Teorem 15-A].

where z^*_{n*} stands for the last but n element of the ω^*-ordered sequence of Z*-points; ω^*-order implies that between any two successive Z*-points no other Z*-point exists. The same applies to the ω -ordering of Z-points. So, even though the points of the X axis are densely ordered, Z*-points and Z-points are not. Because of their successiveness, they can only be traversed in a successive way, one at a time, one after the other. And in such a way that between any two successive Z*-points, or Z-points, a distance greater than zero must always be traversed. Successiveness will play a capital role in the argument that follows.

6 As P passes over the points of the real interval $[0, 1]$ it must traverse the successive Z*-points and the successive Z-Points. In this sense, and being t any instant in $[t_o, t_1]$, let $P^*(t)$ be the number of Z*-points P has traversed just at instant t. And let $P(t)$ be the number of Z-points to be traversed by P at instant t. The discussion that follows examines the evolution of $P^*(t)$ and $P(t)$ as P moves from point 0 to point 1. Both discussions are formalized versions of Zeno's Dichotomy II and I respectively.[5]

7 The strategy of pairing off the Z*-points (or the Z-points) with the successive instants of an strictly increasing infinite sequence of instants was firstly used by Aristotle [2] when trying to solve Zeno's dichotomies. Although Aristotle ended up by rejecting his original strategy, it is still the preferred one to solve both paradoxes. As we will see, however, the successiveness of Z*-points and Z-points leads to a conflicting conclusion.

ZENO'S DICHOTOMY II

8 Let us begin by analyzing the way P passes over the Z*-points. Since the sequence of Z*-points is ω^*-ordered the first element does not exist, and consequently the first n elements do not exist either, for any finite number n. Thus, and taking into account that P is at

[5] See, for instance, [3], [4], [23], [20], [12], [24], [6], [15].

point 0 at t_0 and at point 1 at t_1, we will have:

$$\forall t \in [t_0, t_1] \begin{cases} t = t_0 : & P^*(t) = 0 \\ t > t_0 : & P^*(t) = \aleph_o \end{cases} \quad (3)$$

Therefore, no instant t exists within $[t_0, t_1]$ at which $P^*(t) = n$, whatever be the finite number n, otherwise there would exist the first n elements of an ω^*−ordered sequence. Notice $P^*(t)$ is well defined in the whole interval $[0, 1]$. Thus, equation (3) expresses a dichotomy: $P^*(t)$ can only take two values along the whole interval $[t_0, t_1]$: either 0 or $\aleph_o$.

9 In agreement with 8 and regarding the number of traversed Z*-points, P can only exhibit two states: the state $S^*(0)$ at which it has traversed zero Z*-points, and the state $S^*(\aleph_o)$ at which it has traversed aleph-null Z*-points. Now then, taking into account the successiveness of Z*-points and the fact that between any two successive Z*-points a distance greater than zero always exists, to traverse aleph-null Z*-points, whatsoever they be, means to traverse a distance greater than zero. And, evidently, to traverse a distance greater than zero at the finite velocity v of P means the traversal has to last a time greater than zero.

10 Let d be any real number greater than 0 and consider the real interval $(0, d)$, which contains $2^{\aleph_o}$ points densely ordered. According to the above dichotomy $S^*(0)$-$S^*(\aleph_o)$, at any point x within $(0, d)$ our particle P will already have traversed aleph-null Z*-points. In consequence d is not the distance P must traverse to become $S^*(\aleph_o)$ from $S(0)$. Now then, since d is any real number greater than zero, we must conclude that no real number greater than zero exists such that it could be the distance P must traverse to become $S^*(\aleph_o)$ from $S(0)$. The same conclusion, and for the same reasons, can be deduced for the amount of time P must spend to become $S^*(\aleph_o)$ from $S^*(0)$.

11 In line with 9 and 10, P needs to traverse a distance greater than zero during a time greater than zero to become $S^*(\aleph_o)$ from $S^*(0)$, but neither that distance nor that time can take a value greater than zero. Note this is not a question of indeterminacy but of impossibility: no positive real number exists so that it may be the distance or the time P needs to become $S^*(\aleph_o)$ from $S^*(0)$. None.

ZENO’S DICHOTOMY I

12 We will now examine the way P traverses the Z-points between point 0 and point 1. Being $P(t)$ the number of Z-points to be traversed by P at the precise instant t in $[t_0, t_1]$, that number can only take two values: either $\aleph_o$ or 0. In fact, assume that at any instant t within $[t_0, t_1]$ the number of Z-points to be traversed by P is a finite number $n > 0$. This would imply the impossible existence of the last n points of an $\omega-$ordered sequence of points. Thus, we have a new dichotomy:

$$\forall t \in [t_0, t_1] \begin{cases} t < t_1: & P(t) = \aleph_o \\ t = t_1: & P(t) = 0 \end{cases} \tag{4}$$

Therefore, no instant t exist at which $P(t) = n$, whatever be the finite number n. Notice $P(t)$ is well defined in the whole interval $[0, 1]$. Thus, equation (4) expresses a new dichotomy: $P(t)$ can only take two values: either $\aleph_o$ or 0.

13 In accord with 12 and regarding the number of Z-points to be traversed, P can only exhibit two states: the state $S(\aleph_o)$ at which that number is $\aleph_o$, and the state $S(0)$ at which that number is 0. The number of Z-points to be traversed by P decreases directly from $\aleph_o$ to 0, without finite intermediate states at which only a finite number of Z-points remain to be traversed.

14 Let us now examine the time during which the number of Z-points to be traversed by P decreases. It is evident that P must traverse $\aleph_o$ Z-points in order to become $S(0)$ from $S(\aleph_o)$. Thus, and taking into account the distance between any two successive Z-points is greater than zero, to traverse $\aleph_o$ Z-points means to traverse a distance greater than zero. And being finite the velocity at which P

moves, the time during which the number of Z-points decreases from $\aleph_o$ to 0 must be grater than 0. Being τ any real number greater than, consider the real interval $(0, \tau)$ and any instant τ' within $(0, \tau)$. The number of Z-points to be traversed at instant $t_1 - \tau'$ has not still begin to decrease because that number at $t_1 - \tau'$ is still $\aleph_o$, if not it would be a finite number[6] n and then the impossible last n points of an $\omega-$ordered sequence of points would exist. Consequently, τ, which is any positive real number, is not the time during which the number of Z-points to be traversed by P decreases from $\aleph_o$ to 0.

15 In compliance with 13 and 14, the time during which the number of Z-points to be traversed by P decreases from $\aleph_o$ to 0 must be greater than zero, but cannot take any value greater than zero. A symmetric reasoning now regarding the distance along which the number of Z-points to be traversed by P decreases from $\aleph_o$ to 0, proves that distance, that must be greater than zero, cannot take any value greater than zero either.

Conclusion

16 The transitions from $S^*(0)$ to $S^*(\aleph_o)$ and from $S(\aleph_o)$ to $S(0)$ can only take place along a distance and a time greater than zero but cannot take place along a distance and a time greater than zero because no positive real number exist neither for those distances nor for those times.

17 The above contradictions are direct consequences of the $\omega-$order and ω^*-order which, in turn, are direct consequences of assuming the existence of actual infinite totalities. It is then this assumption, the assumption of the actual infinity, the ultimate cause of both contradictions.

[6] Recall $\aleph_o$ is the smallest infinite cardinal greater than all finite cardinal.

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