

CORRECTING A MINOR ERROR IN CANTOR'S CALCULATION OF THE POWER OF THE CONTINUUM

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ABSTRACT. Cantor's algebraic calculation of the power of the continuum contains an easily repairable error related to Cantor own way of defining the addition of cardinal numbers. The appropriate correction is suggested.

1. THE EXPONENTIATION OF POWERS

Cantor's most significant contribution to the theory of transfinite numbers is, without a doubt, *Beiträge zur Begründung der transfiniten Mengenlehre*¹. A memory of more than 70 pages divided into two parts which appeared in the *Mathematische Annalen* in the years 1895 and 1897 respectively ([4], [5]). *Beiträge's* first six epigraphs are devoted to found the arithmetics of cardinals. Cantor begins by defining the concept of set and the union of disjoint sets, after which he proposes the following definition of power or cardinal number ([6], p. 86):

We call by the name "power" or "cardinal number" of [the set] M the general concept which, by means of our active faculty of thought, arises from the set M when we make abstraction of the nature of its various elements m and of the order in which they are give.

We denote the result of this double act of abstraction, the cardinal number or power of M by \overline{M} .

Cantor continues by defining the concept of equivalence for sets: two sets M and N are said equivalent, symbolically $M \sim N$, if they can be put into a one to one correspondence ([4], p. 86). He then proves that two sets are equivalent if, and only if, they have the same power ([4], pp. 87-88). After extending the notions of "greater than" and "less than" to cardinals numbers, Cantor defines in set theoretical terms the operations of addition and multiplication of cardinals ([4], pp. 91-94). Since multiplication cannot be easily extended to the case of infinitely many factors, Cantor defines the notion of covering in order to define the exponentiation of (finite and transfinite) powers ([6], p. 94):

By a "covering of the set N with elements of the set M ," or more simply, by a "covering of N with M ," we understand a law by which with every element n of N a definite element of M is bound up, where one and the same element of M can come repeatedly into application. The element of M bound up with n is, in a way, a one value function of n , and may be denoted by $f(n)$; it is called a "covering function of n ." The corresponding covering of N will be called $f(N)$.

¹Translated to English by P. E. B. Jourdain in 1915 as *Contributions to the Founding of the Theory of Transfinite Numbers*, [6]

So, if $N = \{a, b, c, d, e\}$ and $M = \{0, 1\}$ the coverings of N are:

$$f(N) = 10110, f'(N) = 00111, f''(N) = 10111\dots \quad (1)$$

The totality of different coverings of N with M , denoted as $(N|M)$, forms a set Cantor called the "covering-set of N with M ":

$$(N|M) = \{f(N), f'(N), f''(N), \dots\} \quad (2)$$

For instance:

$$(\{a, b, c\}|\{0, 1\}) = \{000, 001, 010, 100, 011, 101, 110, 111\} \quad (3)$$

It is immediate that, if N' and M' are two sets such that $N \sim N'$ and $M \sim M'$ it holds $(N|M) \sim (N'|M')$. This equivalence is necessary to prove the cardinality of $(N|M)$ depends exclusively upon the cardinal a of M and b of N , which in turn makes it possible to define the cardinal number a^b as the cardinal of the set $(N|M)$ ([4], p. 95):

$$a^b = \overline{\overline{(N|M)}} \quad (4)$$

It immediately follows from the above definition that ([4], p. 95):

$$((N|M) \cdot (P|M)) \sim ((N, P)|M) \quad (5)$$

$$((P|M) \cdot (P|N)) \sim (P|(M \cdot N)) \quad (6)$$

$$(P|(N|M)) \sim ((P \cdot N)|M) \quad (7)$$

Thus, if $\overline{\overline{N}} = a$, $\overline{\overline{M}} = b$, and $\overline{\overline{P}} = c$, we will have ([4], p. 95):

$$a^b \cdot a^c = a^{b+c} \quad (8)$$

$$a^c \cdot b^c = (a \cdot b)^c \quad (9)$$

$$(a^b)^c = a^{b \cdot c} \quad (10)$$

The main consequence of the concept of covering is, therefore, the possibility of defining the exponentiation of cardinal numbers even for infinite values of the exponent.

Particularly important in Cantor's determination of the power of the continuum is the covering-set of the set \mathbb{N} of all finite cardinals $\{1, 2, 3, \dots\}$ with the set M of two elements $\{0, 1\}$. This covering-set is the set of all binary infinite strings of 1s and 0s:

1110001110001110001101011...
 1000000000000011111110000...
 0001110001110001111111111...
 ⋮

Each of these strings, when preceded by a decimal point, is the binary representation of a real number x in the real interval $[0, 1]$.

2. THE POWER OF THE CONTINUUM

In the year 1872 Cantor published a paper on the continuity and infiniteness of the set \mathbb{R} of real numbers [1]. In this paper, Cantor considered axiomatic the one to one correspondence between the real line points (linear continuum) and the real numbers. Two years after, in 1874, he gave his first proof on the non enumerable nature of the set \mathbb{R} of real numbers [1] (his second proof -the diagonal method- on the existence of non denumerable sets was published in 1891 [3]). In 1878 he proved the equivalence between linear and n-dimensional continuums [2]. All these results were directly or indirectly necessary to prove the power of the continuum is the transfinite cardinal 2^{\aleph_0} ([6], p. 96):

We see how pregnant and far-reaching these simple formulæ extend to powers are by the following example. If we denote the power of the linear continuum X (that is the totality X of real numbers x such that $x \geq$ and ≤ 1) by c , we easily see that it may be represented by, among others, the formula:

$$c = 2^{\aleph_0} \tag{11}$$

where §6 gives the meaning² of 2^{\aleph_0} .. In fact, by (4), 2^{\aleph_0} is the power of all representations

$$x = \frac{f(1)}{2} + \frac{f(2)}{2^2} + \dots + \frac{f(\nu)}{2^\nu} + \dots \text{ (where } f(\nu) = 0 \text{ or } 1 \text{)} \tag{12}$$

of the numbers x in the binary system. If we pay attention to the fact that every number x is only represented once, with the exception of the numbers $x = \frac{2\nu+1}{2^\nu} < 1$, which are represented twice over, we have, if we denote the "enumerable" totality of the latter by $\{s_\nu\}$,

$$2^{\aleph_0} = \overline{\overline{\{\{s_\nu\}, X\}}} \tag{13}$$

If we take away from X any "enumerable" set $\{t_\nu\}$ and denote the remainder by X_1 , we have:

$$X = (\{t_\nu\}, X_1) = (\{t_{2\nu-1}\}, \{t_{2\nu}\}, X_1) \tag{14}$$

$$(\{s_\nu\}, X) = (\{s_\nu\}, \{t_\nu\}, X_1) \tag{15}$$

²In §6 Cantor defines \aleph_0 as the smallest transfinite cardinal number: the cardinal of the set of all finite cardinals: $\aleph_0 = \overline{\{\nu\}}$

$$\{t_{2\nu-1}\} \sim \{s_\nu\}, \{t_{2\nu}\} \sim \{s_\nu\}, X_1 \sim X_1 \quad (16)$$

so

$$X \sim (\{s_\nu\}, X) \quad (17)$$

and thus

$$2^{\aleph_0} = \overline{\overline{X}} = c \quad (18)$$

According to Cantor's notation, (M, N) is the union of two sets M and N which have no common elements ([4], p. 85). The union of disjoint sets is essential in Cantor's definitions of arithmetic operations. So, if $\overline{M} = a$ and $\overline{N} = b$, the sum of the cardinals a and b is given by ([4], p. 91):

$$a + b = \overline{\overline{(M, N)}} \quad (19)$$

being $M \cap N = \emptyset$. It is therefore clear the meaning of the above equation (13)

$$2^{\aleph_0} = \overline{\overline{(\{s_\nu\}, X)}} \quad (20)$$

Although being $\{s_\nu\}$ the enumerable totality of real numbers $x = \frac{2\nu+1}{2^\mu} < 1$, which have two binary representations ([4], p. 96), and X the set of all real numbers in $[0, 1]$ ([4], p. 96), it is also clear that $\{s_\nu\} \subset X$, so that $\{s_\nu\} \cap X \neq \emptyset$. This little difficulty is easily solved by redefining the sets involved in the proof. In fact, let B be the set of all binary infinite strings of 0s and 1s (the covering-set of \mathbb{N} by $\{0, 1\}$). By definition, we have

$$\overline{\overline{B}} = 2^{\aleph_0} \quad (21)$$

The set B can be divided into two disjoint sets: the set B_X and the set B_S . The set B_X is the set of all binary strings representing all real numbers of the set $X = [0, 1]$ except the strings of the second binary expressions of all X 's elements which have two binary expressions (the first one ending by an infinite string of 0s, and the second by an infinite string of 1s). The set B_S is just the denumerable set of all those second binary strings. Evidently, we will have:

$$B = B_X \cup B_S \quad (22)$$

and:

$$B_X \sim X \sim \mathbb{R} \quad (23)$$

being B_X , X and \mathbb{R} non denumerable. From this point we only have to follow Cantor's argument. Let T be any denumerable subset of B_X and let B'_X be the complement of T with respect to B_X , i.e. $B'_X = B_X - T$, we can write:

$$B_X = T \cup B'_X \quad (24)$$

being $T \cap B'_X = \emptyset$. Since T is denumerable its elements can be indexed by the totality of natural numbers. Consequently we can consider two disjoint denumerable subsets T_E and T_O , whose elements are respectively indexed by the even and the odd natural numbers. Equation (24) can then be rewritten as:

$$B_X = T_E \cup T_O \cup B'_X \quad (25)$$

where $T_E \cup T_O = T$; $T_E \cap T_O = \emptyset$. From (24) we also get:

$$B_S \cup B_X = B_S \cup T \cup B'_X \quad (26)$$

and being

$$T_E \sim B_S \quad (T_E \text{ and } B_S \text{ are denumerable}) \quad (27)$$

$$T_O \sim T \quad (T_O \text{ and } T \text{ are denumerable}) \quad (28)$$

$$B'_X \sim B'_X \quad (\text{Every set is equivalent to itself}) \quad (29)$$

$$(30)$$

we have:

$$T_E \cup T_O \cup B'_X \sim B_S \cup T \cup B'_X \quad (31)$$

and then, in accordance with (25) and (26):

$$B_X \sim B_S \cup B_X \quad (32)$$

and taking into account that $B_S \cup B_X = B$, we get:

$$B_X \sim B \quad (33)$$

and then

$$\overline{\overline{B_X}} = \overline{\overline{B}} = 2^{\aleph_0} \quad (34)$$

Finally, being $B_X \sim X \sim \mathbb{R}$, we can write:

$$\overline{\overline{X}} = \overline{\overline{\mathbb{R}}} = 2^{\aleph_0} \quad (35)$$

which proves that, in fact, the power of the continuum is the transfinite cardinal 2^{\aleph_0} .

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