1.-Cantor versus Cantor

THE n-TH DECIMAL THEOREM

1 Cantor's diagonal argument makes use of an hypothetical table T that is assumed to contain all real numbers within the real interval $(0,1)$. That table can be easily redefined in order to ensure it contains at least all rational numbers within $(0,1)$. In these conditions, could the rows of T be reordered so that a rational antidiagonal can be defined? In that case, and for the same reason as in Cantor original proof, the set of rational numbers would be non denumerable. And then we would have a contradiction since Cantor also proved the set of rational numbers is denumerable. Should, therefore, Cantor's diagonal argument be suspended until it be proved the impossibility of such a reordering? Is that reordering possible? The discussion that follows addresses both questions.

2 We will begin by proving an elementary result on the decimal expansion of rational numbers.[1] For this, let M be the set of all real numbers within the real interval $(0,1)$ expressed in decimal notation and completed, in the cases of finitely many decimal digits, with infinitely many 0's in the right side of their decimal expansions, so in the place of 0.25 we will write $0.25000\ldots$. The subset of all rational numbers in M will be denoted by M_Q.

3 Let us prove the following:

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[1] It could easily be extended to irrational numbers, although we will not do it here.

> Theorem of the nth decimal.-*For every natural number n there are infinitely many different elements in M_Q with the same decimal digit d_n in the same nth position of its decimal expansion.*

Proof.-Let d_n be any decimal digit (0, 1, 2, . . . 9) and consider any element r_0 in M_Q of the form:

$$r_0 = 0.d_1d_2 \dots d_n000 \dots \tag{1}$$

where each d_i is any decimal digit. From r_0 we define the sequence of rational numbers:

$$\begin{aligned}
&r_1 = 0.d_1d_2 \dots d_n1000 \dots && (2)\\
&r_2 = 0.d_1d_2 \dots d_n11000 \dots && (3)\\
&r_3 = 0.d_1d_2 \dots d_n111000 \dots && (4)\\
&\dots\\
&r_k = 0.d_1d_2 \dots d_n1 \overset{(k)}{\dots} 1000 \dots && (5)\\
&\dots
\end{aligned}$$

The one to one correspondence f between $\mathbb{N}$ (the set of natural numbers) and M_Q defined according to:

$$f(k) = r_k, \forall k \in \mathbb{N} \tag{6}$$

proves the existence of a denumerable subset $f(\mathbb{N})$ of M_Q, all of whose elements have the same decimal digit d_n in the same nth position of its decimal expansion.

CANTOR’S DIAGONAL ARGUMENT

4 Cantor’s set M is the union of two disjoint sets: the denumerable set M_Q of all rational numbers in $(0, 1)$ and the set M_I of all irrationals in the same interval $(0, 1)$. Assume, as Cantor did in 1891 [2], M were denumerable. In those conditions, it is evident that, being M_I infinite, it will also be denumerable, otherwise its superset M could not be denumerable. Let then g be a bijection between $\mathbb{N}$ and M_Q, and h a bijection between $\mathbb{N}$ and M_I. From g and h we define a one to one correspondence or bijection f between $\mathbb{N}$ and M according to:

$$\left.\begin{aligned} f(2n-1) &= g(n) \\ f(2n) &= h(n) \end{aligned}\right\} \forall n \in \mathbb{N} \tag{7}$$

We can therefore consider the $\omega-$ordered table T whose successive rows r_1, r_2, r_3 ... are just $f(1)$, $f(2)$, $f(3)$ .... By construction, and being M_Q (supposedly) denumerable, T contains a denumerable subtable with *all* rational numbers in $(0,1)$.

5 The diagonal of T is a real number $D = 0.d_{11}d_{22}d_{33}\dots$ whose nth decimal digit d_{nn} is the nth decimal digit of the nth row r_n of T. Cantor successfully proved [2] the existence of another real number in M derived from D, the antidiagonal D^-, which cannot be in T. In consequence, M cannot be denumerable as was assumed to be (Cantor's diagonal argument, an impeccable Modus Tollens (MT)[2] [2]).

6 Since D^- is a real number in $(0,1)$, it will be either rational or irrational. But if it were rational then, and for the same reason as in the case of M, the subset $M_{\mathbb{Q}}$ of all rational numbers in M would also be non countable. The problem here is that Cantor had already proved the set $\mathbb{Q}$ of all rational numbers, and therefore $M_{\mathbb{Q}}$, is countable [1].

7 According to 6, if it were possible to reorder the rows of T in such a way that a *rational* antidiagonal could be defined, then we would have two contradictory results: the set $\mathbb{Q}$ of rational numbers would and would not be denumerable. Both results could be considered as proved by Cantor, although the second one only as an unexpected (and so far unknown) consequence of his famous diagonal method.

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[2] The critiques of Cantor's diagonal argument are invariably related to constructionist aspects which are not pertinent with the formal structure of Cantor's demonstration.

Accordingly, we can state the following:

> Conclusion 7.-*Cantor's diagonal argument and all its formal consequences should be suspended until it be proved the impossibility of reordering the rows of T in such a way that a rational antidiagonal can be defined.*

RATIONAL ANTIDIAGONALS

8 We will now examine the possibilities and consequences of reordering the rows of T in the sense indicated in 7.

9 A formal consequence of the existence of complete infinite totalities (actual infinity) is the existence of ω-ordered sequences [3], [4, Theoreme 15-A]. In an ω-ordered sequence, as our table T, *every* element -whatever it be- will always be preceded by a finite number of elements and followed by an infinite number of elements. We will see now a conflicting consequence of this immense and suspicious asymmetry.

10 We will begin by defining the concept of D-modular row. A row r_i of T will be said n-modular if its nth decimal digit is (n mod 10). This means that a row is, for instance, 2348-modular if its 2348th decimal digit is 8; or that it is 453-modular if its 453th decimal digit is 3. If a row r_n is n-modular (being n in n-modular the same number as n in r_n) it will be said *D-modular*. For instance, the rows:

$$r_1 = 0.\mathbf{1}007647464749943400034577774413\ldots \tag{8}$$

$$r_2 = 0.2\mathbf{2}000456677789430000000000000000\ldots \tag{9}$$

$$r_3 = 0.00\mathbf{3}0000000000000000000000000000\ldots \tag{10}$$

$$r_9 = 0.11122233\mathbf{9}0000004340666666666333\ldots \tag{11}$$

$$r_{13} = 0.123456789000\mathbf{3}000567585843456931\ldots \tag{12}$$

are all of them D-modular.

11 Consider now the following permutation **P** of the rows $\langle r_n \rangle_{n \in \mathbb{N}}$ of table T:

For each successive row r_i in T:

- If r_i is D-modular then let it unchanged.
- If r_i is not D-modular then exchange it with any following i-modular row $r_{j, j>i}$, provided that at least one of the following rows be i-modular.[3]

The exchange of a non D-modular row r_i with a *following i-modular row* will be referred to as *P-exchange*.

12 Notice that, thanks to the condition $j > i$ (in $r_{j, j>i}$), once a non-D-modular row r_i has been P-exchanged, it becomes D-modular and will remain D-modular and unaffected by the subsequent P-exchanges. And notice also the successive P-exchanges do not alter the $\omega-$ordering of table T: **P** does not modify the ordering of the $\omega-$ordered set $\mathbb{N}$ of indexes, but the real numbers indexed by the same successive indexes. Or in other words, P-exchanges do not exchange the indexes but the real numbers indexed by them. It exchanges the content of the successive T's rows, but not the successiveness of T's rows.

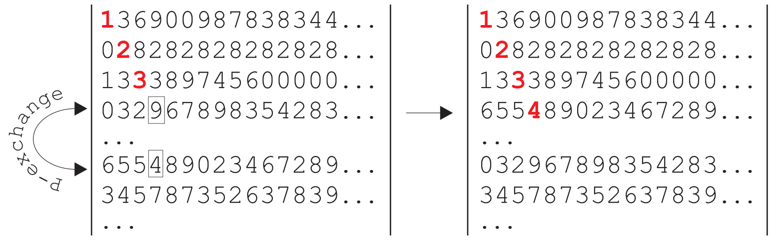

Figure 1.1: Left: r_4 before being P-exchanged. Right: Once P-exchanged, r_4 is a D-modular row.

13 It is immediate to prove that all T's rows become D-modular as a consequence of permutation **P**. In fact, let us assume that a row r_n

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[3] Replace r_i with r_j and r_j with r_i.

did not become D-modular as a consequence of **P**. This means that r_n could not be P-exchanged with a following n-modular row. Now then, all n-modular rows have the same digit (n mod 10) in the same nth position of its decimal expansion, and according to the theorem 3 of the nth decimal, there are infinitely many rational numbers with the same digit in the same position of its decimal expansion, whatsoever be the digit and the position. Accordingly, since n is finite, the row r_n is preceded by a finite number and followed by an infinite number of n-modular rows. Any of these infinitely many succeeding n-modular rows had to be P-exchanged by r_n. It is therefore impossible for r_n not to be D-modular. In consequence (Modus Tollens), each and every row r_n becomes D-modular as a consequence of **P**.

14 It is worth noting the result proved in 13 is a formal consequence of both theorem 3 of the nth decimal and the fact that *every row* r_n of T is always preceded by a finite number of n-modular rows and followed by an infinite number of n-modular rows. This immense asymmetry is an inevitable side effect of $\omega-$order, which, as Cantor proved [4, Theorem 15-A], derives from assuming the existence of the set of all finite cardinals (natural numbers) as a complete totality (hypothesis of the actual infinity subsumed within the Axiom of Infinity).

15 Let us remark the formal structure of proof 13 in order to avoid unnecessary discussions. Consider the following two propositions q_1 and q_2 about permutation **P**:

- q_1: Not all rows become D-modular as a consequence of permutation **P**.
- q_2: At least a row r_k could not be P-exchanged.

It is quite clear that q_1 implies q_2: if not all rows become D-modular then at least a row r_k could not be P-exchanged. Now then, being k finite and taking into account the theorem of the nth decimal, there are infinitely many k-modular rows $r_{n,\,n>k}$ following r_k, therefore some of them had to be P-exchanged by r_k. In consequence proposition q_2 is false and then so will be q_1. In symbols:

$$(q_1 \Rightarrow q_2) \wedge (\neg q_2) \Rightarrow \neg q_1 \tag{13}$$

It is quite clear then that, as in the case of Cantor's diagonal argument, the above proof is also a simple Modus Tollens (see final remark).

16 Let T_p be the table resulting from permutation $\mathbf{P}$. Since all its rows are D-modular, its diagonal D will be the rational number $0.\overline{1234567890}$. It is now immediate to define infinitely many rational antidiagonals from D. Indeed, let us denote by p_o the period 1234567890 of the diagonal D. We are interested in periods of ten digits none of which coincide in position with the digits of p_o, as is the case, for instance, of 0123456789 or 4545454545 ($= \widehat{45}$). The number of those periods is 9^{10}. Among them, let us choose the above two examples and denote them by p_1 and p_2 respectively ($p_1 = 0123456789$; $p_2 = 4545454545$). Now we define the sequence of rational antidiagonals $\langle A_n \rangle_{n \in \mathbb{N}}$ by:

$$\forall n \in \mathbf{N}: \; A_n = 0.p_1 p_1 \overset{n}{\ldots} p_1 \widehat{p}_2 \tag{14}$$

whose elements cannot be in T_p for the same reason as in Cantor's antidiagonal. Since all of them are rational numbers, we must conclude that M_Q and its superset $\mathbb{Q}$ are both non denumerable.

17 Permutation $\mathbf{P}$ allows us to develop other arguments whose conclusions also suggest the inconsistency of the hypothesis of the actual infinity. For instance, it is clear that rows as $0.\widehat{21}$, and many others, can never become D-modular, and then we would have to admit the absurdity that $\mathbf{P}$ makes all of them disappear from the table. In fact, let n be any natural number and assumes that, for instance, $0.\widehat{21}$ is the nth row of T_p. Since n is finite, $0.\widehat{21}$ will be preceded by a finite number of n-modular rows and followed by an infinite number of n-modular rows, according to theorem 3 of the nth decimal. In consequence $0.\widehat{21}$, that is not n-modular,[4] was P-exchanged with any of those n-modular rows, and then it cannot be the nth row of T_p. Thus, and being r_n any row of T, we must conclude $0.\widehat{21}$ has disappeared from the table!

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[4] For each nth decimal digit of $0.\widehat{21}$ it holds (n mod 10) = 2 if n is odd, or (n mod 10) = 1 if it is even.

18 The above absurdity 17 is the sort of things you can expect from a list in which *each and every* element has *finitely many* predecessors and *infinitely many* successors. A list in which, in spite of having infinitely many successive elements, it is impossible to reach an element with infinitely many predecessors (what, evidently, makes the above arguments possible). A list, in short, that is simultaneously complete (as the hypothesis of the actual infinity requires) and uncompletable (because no last element completes it).

19 Permutation **P** can even be considered as a case of supertask (hypercomputation). In fact, let $\langle t_n \rangle_{n \in \mathbb{N}}$ be an $\omega-$ordered increasing sequence of instants within a finite interval of time (t_a, t_b), being t_b the limit of the sequence. Assume that **P** operates on each row r_i just at the precise instant t_i of $\langle t_n \rangle_{n \in \mathbb{N}}$. Consequently, r_i will remain unchanged if it is D-modular (or if it is not D-modular and can not be P-exchanged) or it will be P-exchanged with any of the following i-modular rows. At t_b permutation **P** will have been applied to every row of T as the one to one correspondence $f(t_i) = r_i$ proves.

20 Assume that at t_b, once accomplished the hypercomputation **P**, T_p contains a row r_n which is not D-modular. This row, whatsoever it be, will be preceded by a finite number of rows and followed by an infinite number of rows, infinitely many of which are n-modular, and then P-exchangeable with r_n. Thus, either r_n is D-modular in T_p, or it has magically disappeared from the table!

21 To be simultaneously complete and uncompletable, as is the case of any $\omega-$ordered object, could be, after all, contradictory.

A FINAL REMARK

22 Let me end by recalling that an argument cannot be refuted by another different argument. In W. Hodges words: [5, p. 4]

> How does anybody get into a state of mind where they persuade themselves that you can criticize an argument by suggesting a different argument which doesn't reach

the same conclusion?

This inadmissible strategy is frequently used in discussions related to infinity, for instance to refute Cantor's arguments on the uncountable nature of real numbers. However, to refute an argument means to indicate where and why that argument fails. If two arguments lead to contradictory conclusions, they simply are proving the existence of a contradiction.

Bibliography

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[5] Wilfrid Hodges, *An Editor Recalls some Hopeless Papers*, The Bulletin of Symbolic Logic **4** (1998), no. 1, 1–16.