1.-Cantor's 1874-argument revisited

INTRODUCTION

1 This chapter examines the conditions under which Cantor's 1874 argument on the uncountable nature of real numbers could also be applied to rational numbers. It will necessary, therefore, to prove those conditions can never be fulfilled in order to ensure the impossibility of a contradiction on the cardinality of the set of rational numbers, that was proved to be denumerable by Cantor himself [3]. A short rational variant of Cantor's argument is also included.

CANTOR'S 1874-ARGUMENT

2 This section explains in detail the first Cantor's proof of the uncountable nature of the set $\mathbb{R}$ of real numbers, published in the year 1874 in a short paper that also included a proof of the countable nature of the set of algebraic numbers and then of the set of rational numbers [3], (French edition [4], Spanish edition [5]).

3 Assume the set $\mathbb{R}$ were denumerable. In those conditions there would be a one to one correspondence f between the set $\mathbb{N}$ of natural numbers and $\mathbb{R}$. Consequently, the elements of $\mathbb{R}$ could be $\omega-$ordered[1] by f as:

$$r_1,\ r_2,\ r_3,\ \dots \tag{1}$$

being $r_i = f(i), \forall i \in \mathbb{N}$. Obviously, the sequence $\langle r_n \rangle_{n\in\mathbb{N}}$ defined by f *would contain* all real numbers if $\mathbb{R}$ were really denumerable.

4 Consider now any real interval (a, b). Cantor's 1874-argument consists in proving the existence of a real number η in (a, b) which is not in the $\omega-$ordered sequence $\langle r_n \rangle_{n\in\mathbb{N}}$. The existence of η would prove that $\langle r_n \rangle_{n\in\mathbb{N}}$ does not contain all real numbers and, therefore, that the initial assumption on the countable nature of $\mathbb{R}$ is false. The proof goes as follows.

5 Starting from r_1, find the *first two* elements of $\langle r_n \rangle_{n\in\mathbb{N}}$ within (a, b). Denote the smaller of them by a_1 and the greater by b_1. Define the real interval (a_1, b_1) (see Figure 1.1).

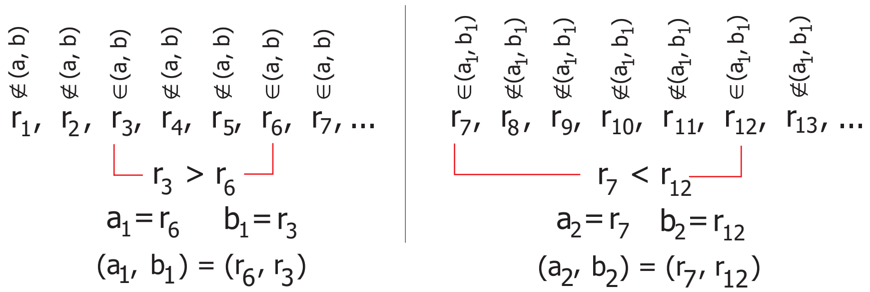

Figure 1.1: Definition of the two first intervals (a_1, b_1), (a_2, b_2).

6 Starting from r_1, find the *first two* elements of $\langle r_n \rangle_{n\in\mathbb{N}}$ within (a_1, b_1). Denote the smaller of them by a_2 and the greater by b_2. Define the real interval (a_2, b_2). Evidently it holds:

$$(a_1, b_1) \supset (a_2, b_2) \tag{2}$$

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[1] As is well known, in the $\omega-$ordered sequences there exists a first element and each element has an immediate predecessor, except the first one, and an immediate successor so that no last element exists.

7 Starting from r_1, find the *first two* elements of $\langle r_n \rangle_{n\in\mathbb{N}}$ within (a_2, b_2). Denote the smaller of them by a_3 and the greater by b_3. Define the real interval (a_3, b_3). Evidently it holds:

$$(a_1, b_1) \supset (a_2, b_2) \supset (a_3, b_3). \tag{3}$$

8 The continuation of the above procedure (R from now on) defines a sequence of real nested intervals (R-intervals):

$$(a_1,\ b_1) \supset (a_2,\ b_2) \supset (a_3,\ b_3) \supset \ldots \tag{4}$$

whose left endpoints a_1, a_2, a_3,... form a strictly increasing sequence of real numbers, and whose right endpoints b_1, b_2, b_3,... form a strictly decreasing sequence also of real numbers, being every element of the first sequence smaller than every element of the second one.

9 From the $\omega-$order of $\langle r_n \rangle_{n\in\mathbb{N}}$ and the ordered way R defines the successive R-intervals (starting from r_1 find the first two elements. . .), it immediately follows that if r_n defines an endpoint a_i or b_i, then it must hold $i \leq n$. In consequence, if r_n is any element of $\langle r_n \rangle_{n\in\mathbb{N}}$, it will not belong to the successive intervals:

$$(a_n, b_n),\ (a_{n+1}, b_{n+1}),\ (a_{n+2}, b_{n+2}),\ \ldots \tag{5}$$

10 The number of R-intervals will be be finite or infinite, and both possibilities have to be considered. Assume in the first place the number of R-intervals is finite.[2] In this case there would be a last interval[3] (a_n, b_n) in the sequence of intervals. This last interval would contain, at best, one element r_v of $\langle r_n \rangle_{n\in\mathbb{N}}$, otherwise it would be possible to define at least one new real interval $(a_{n+1},\ b_{n+1})$. Let, therefore, η be any element within (a_n, b_n), different from r_v if r_v does exist. Evidently η is a real number within (a, b) which does not belong to the sequence $\langle r_n \rangle_{n\in\mathbb{N}}$. Consequently, the sequence $\langle r_n \rangle_{n\in\mathbb{N}}$ does not contain

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[2] Including the case that R defines no interval.

[3] Or the whole interval (a, b) in the case that R defines no interval.

all real numbers, and then the initial assumption on the countable nature of $\mathbb{R}$ must be false.

11 Consider now the number of R-intervals is infinite[4]. Since the sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ is strictly increasing and upper bounded by every element of $\langle b_n \rangle_{n\in\mathbb{N}}$, the limit L_a of $\langle a_n \rangle_{n\in\mathbb{N}}$ does exist. On its part, the sequence $\langle b_n \rangle_{n\in\mathbb{N}}$ is strictly decreasing and lower bounded by every element of $\langle a_n \rangle_{n\in\mathbb{N}}$, in consequence the limit L_b of this sequence also exists. Taking into account that every a_i is less than every b_i it must hold: $L_a \leq L_b$.

12 Assume that $L_a < L_b$. In this case any of the infinitely many elements within the real interval (L_a, L_b) is a real number within (a, b) which does not belong to the sequence $\langle r_n \rangle_{n\in\mathbb{N}}$, and then a proof of the falseness of the initial hypothesis on the countable nature of $\mathbb{R}$.

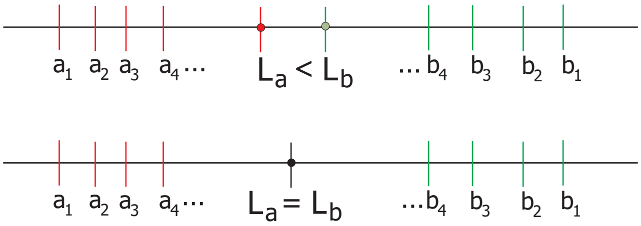

Figure 1.2: Convergence of $\langle a_n \rangle_{n\in\mathbb{N}}$ and $\langle b_n \rangle_{n\in\mathbb{N}}$.

13 Finally, assume that $L_a = L_b = L$. It is immediate to prove that L is a real number within (a, b) which is not in $\langle r_n \rangle_{n\in\mathbb{N}}$. In fact, assume that L is an element r_v of $\langle r_n \rangle_{n\in\mathbb{N}}$. According to 9, r_v does not belong to the successive intervals:

$$(a_v, b_v),\ (a_{v+1}, b_{v+1}),\ (a_{v+2}, b_{v+2}), \dots, \tag{6}$$

while L belongs to all of them. Therefore, L cannot be r_v. The limit L is a real number in (a, b) which is not in $\langle r_n \rangle_{n\in\mathbb{N}}$, and then a proof of

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[4] Note this case implies the completion of a procedure of infinitely many successive steps.

the falseness of the initial assumption on the countable nature of $\mathbb{R}$.

$\mathbb{Q}$-VERSION OF CANTOR’S 1874-ARGUMENT

14 This section develops an argument which is identical to the previous one, except in that it applies to the set $\mathbb{Q}$ of rational numbers.

15 Assume the set $\mathbb{Q}$ of rational numbers were denumerable. In these conditions there would be a one to one correspondence f between the set $\mathbb{N}$ of natural numbers and $\mathbb{Q}$ so that the elements of $\mathbb{Q}$ could be $\omega-$ordered by f as:

$$q_1,\ q_2,\ q_3,\ \dots \tag{7}$$

being $q_i = f(i), \forall i \in \mathbb{N}$. Obviously, the sequence $\langle q_n \rangle_{n \in \mathbb{N}}$ defined by f *would contain* all rational numbers if $\mathbb{Q}$ were really denumerable.

16 Consider any rational interval (a, b). Starting from q_1, find the *first two* elements of $\langle q_n \rangle_{n \in \mathbb{N}}$ within (a, b). Denote the smaller of them by a_1 and the greater by b_1. Define the rational interval (a_1, b_1).

17 Starting from q_1, find the *first two* elements of $\langle q_n \rangle_{n \in \mathbb{N}}$ within (a_1, b_1). Denote the smaller of them by a_2 and the greater by b_2. Define the rational interval (a_2, b_2). Evidently it holds:

$$(a_1, b_1) \supset (a_2, b_2) \tag{8}$$

18 Starting from q_1, find the *first two* elements of $\langle q_n \rangle_{n \in \mathbb{N}}$ within (a_2, b_2). Denote the smaller of them by a_3 and the greater by b_3. Define the rational interval (a_3, b_3). Evidently it holds:

$$(a_1, b_1) \supset (a_2, b_2) \supset (a_3, b_3). \tag{9}$$

19 The continuation of the above procedure (Q from now on) defines a sequence of rational nested intervals (Q-intervals):

$$(a_1,\ b_1) \supset (a_2,\ b_2) \supset (a_3,\ b_3) \supset \dots \tag{10}$$

whose left endpoints a_1, a_2, a_3,... form a strictly increasing sequence of rational numbers, and whose right endpoints b_1, b_2, b_3,... form a strictly decreasing sequence of rational numbers, being every element of the first sequence smaller than every element of the second one.

20 From the $\omega-$order of $\langle q_n \rangle_{n\in\mathbb{N}}$ and the ordered way Q defines the successive Q-intervals (starting from q_1 find the first two elements...), it immediately follows that if q_n defines an endpoint a_i or b_i, then it must hold $i \leq n$. In consequence, if q_n is any element in $\langle q_n \rangle_{n\in\mathbb{N}}$, it will not belong to the successive intervals:

$$(a_n, b_n),\ (a_{n+1}, b_{n+1}),\ (a_{n+2}, b_{n+2}), \ldots \tag{11}$$

21 The number of Q-intervals will be be finite or infinite, and both possibilities have to be considered. Assume in the first place the number of Q-intervals is finite[5]. In this case there would be a last rational interval[6] (a_n, b_n) in the sequence of intervals. This last interval would contain, at best, one element q_v of $\langle q_n \rangle_{n\in\mathbb{N}}$, otherwise it would be possible to define at least one new rational interval (a_{n+1}, b_{n+1}). Let, therefore, η be any element within (a_n, b_n), different from q_v in the case that q_v does exist. Evidently η is a rational number within (a, b) which does not belong to the sequence $\langle q_n \rangle_{n\in\mathbb{N}}$. Consequently, the sequence $\langle q_n \rangle_{n\in\mathbb{N}}$ does not contain all rational numbers, and then our initial assumption on the countable nature of $\mathbb{Q}$ must be false.

22 Consider now the number of Q-intervals is infinite.[7] Since the sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ is strictly increasing and upper bounded by every element of $\langle b_n \rangle_{n\in\mathbb{N}}$, the *real* limit L_a of $\langle a_n \rangle_{n\in\mathbb{N}}$ does exist. On its part, the sequence $\langle b_n \rangle_{n\in\mathbb{N}}$ is strictly decreasing and lower bounded by every element of $\langle a_n \rangle_{n\in\mathbb{N}}$, in consequence the *real* limit L_b of this sequence also exists. Taking into account that every a_i is less than every b_i it must hold: $L_a \leq L_b$, being L_a and L_b two real (rational or irrational) numbers.

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[5] Including the case that Q defines no interval.

[6] Or the whole interval (a, b) in the case that Q defines no interval.

[7] This case implies the completion of a procedure of infinitely many successive steps.

23 Assume that $L_a < L_b$. In this case, any of the infinitely many rationals within the real interval (L_a, L_b) is a rational number within (a, b) which does not belong to the sequence $\langle q_n \rangle_{n \in \mathbb{N}}$, and then a proof of the falseness of the initial hypothesis on the countable nature of $\mathbb{Q}$.

24 Finally, assume that $L_a = L_b = L$. It is immediate that L is a *real* number within the *real interval* (a, b) which is not in $\langle q_n \rangle_{n \in \mathbb{N}}$. In fact, if L is irrational then it is clear that it is not in $\langle q_n \rangle_{n \in \mathbb{N}}$; assume then L is rational, and assume also it is an element q_v of $\langle q_n \rangle_{n \in \mathbb{N}}$. According to 20, q_v does not belong to the successive intervals:

$$(a_v, b_v),\ (a_{v+1}, b_{v+1}),\ (a_{v+2}, b_{v+2}),\ \ldots \tag{12}$$

while L belongs to all of them. Therefore, L cannot be q_v. The limit L is a real number (rational or irrational) in the real interval (a, b) which is not in $\langle q_n \rangle_{n \in \mathbb{N}}$. Thus, if L were rational then our initial assumption on the countable nature of $\mathbb{Q}$ would be false.

25 We have just proved the alternatives of Cantor 1874-argument on the cardinality of the real numbers also applies to the set $\mathbb{Q}$ of rational numbers, except the last one, that applies only if the common limit of the sequences of left and right rational endpoints of the Q-intervals is a rational number.

26 Evidently, If Cantor's 1874-argument could be extended to the rational numbers we would have a contradiction: the set $\mathbb{Q}$ would and would not be denumerable. In consequence, and in order to ensure the impossibility of that contradiction, we must prove that whatsoever be the rational interval (a, b) and the reordering of $\langle q_n \rangle_{n \in \mathbb{N}}$, the number of Q-intervals can never be finite and the sequences of endpoints $\langle a_n \rangle_{n \in \mathbb{N}}$ and $\langle b_n \rangle_{n \in \mathbb{N}}$ have always a common *irrational* limit. Until then, the consistency of transfinite set theory will be at stake.

A RATIONAL VARIANT OF CANTOR'S 1874 ARGUMENT

27 The argument that follows is a variant of the above Cantor's first proof of the uncountable nature of the set of real numbers.

28 Since the set $\mathbb{Q}$ of rational numbers is denumerable we can consider a one to one correspondence f between this set and the set $\mathbb{N}$ of natural numbers. Let $\langle q_n \rangle_{n \in \mathbb{N}}$ be the ω−ordered sequence of rational numbers defined by:

$$q_i = f(i), \ \forall i \in \mathbb{N} \tag{13}$$

Obviously $\langle q_n \rangle_{n \in \mathbb{N}}$ contains all rational numbers.

29 Let x be a rational variable whose initial value is b, the right endpoint of any rational interval (a, b); and $\langle q_n \rangle_{n \in \mathbb{N}}$ the sequence of rational numbers defined by (13). Now consider the following sequence of x redefinitions:

$$\begin{cases} i = 1, 2, 3, \dots \\ q_i \in (a, b) \wedge q_i < x \Rightarrow x = q_i \end{cases} \tag{14}$$

that compares x with the successive elements of $\langle q_n \rangle_{n \in \mathbb{N}}$ that belong to (a, b), and redefines x as the compared element each time the compared element is less than the current value of x.

30 Unnecessary as it may seem, we will impose the following restriction to the successive definitions (14):

> Restriction 30.-*Each successive definition (14) will be carried out if, and only if, x results defined as a rational number within $(a, b]$.*

We will prove now that for any natural number v the first v definitions (14) can be carried out.

31 Since q_1 is a well defined rational number we will know if it is within (a, b) and if it less than x, whose current value is b. Thus, the first definition (14) can be carried out because it defines x as b or as q_1 if q_1 is in (a, b) and $q_1 < b$. Assume that, being n any natural number, the first n definitions (14) can be carried out, and let x_c be the corresponding current value of x once performed the first n definitions (14). Since q_{n+1} is a well defined rational number, we will know if it is within (a, b) and if it is less than x_c. Thus the $(n + 1)$th definition can also be carried out because it defines x as x_c, or as q_{n+1} if q_{n+1} is in (a, b) and $q_{n+1} < x_c$. We have just proved that the first definition (14) can be carried out, and that if for any natural number n the first n definitions (14) can be carried out, then the first $(n + 1)$ definitions (14) can also be carried out. Thus, for any natural number v, the first v definitions (14) can be carried out.

32 Assume that while the successive definitions (14) can be carried out, they are carried out. The value of x once performed all possible definitions (14),[8] whatsoever be the finite or infinite number of times it has been redefined, will be a rational number within the interval $(a, b]$ just because *it was always defined, including the initial definition, as a rational number within* $(a, b]$, *and only as a rational number within* $(a, b]$. Thus, we can state:

- Whatever be the current value x^* of x once performed all possible redefinitions (14), it will be a rational number within the rational interval $(a, b]$.

33 Consider the rational interval (a, x^*) and any element η within (a, x^*), as for instance $1/2(a + x^*)$. It is quite clear that $\eta \in (a, b)$ and $\eta < x^*$. We will prove η cannot belong to $\langle q_n \rangle_{n\in\mathbb{N}}$. In fact, assume η belongs to $\langle q_n \rangle_{n\in\mathbb{N}}$. In such a case there will be an element q_v in $\langle q_n \rangle_{n\in\mathbb{N}}$ such that $q_v = \eta$, and being η in (a, x^*), we will have $q_v \in (a, x^*)$, and therefore $q_v < x^*$. But this is impossible because:

1. The index v of q_v is a natural number.
2. According to 31, for each natural number v, it is possible to carry out the first v definitions (14)
3. All possible definitions (14) have been carried out.
4. At least the first v definitions (14) have been carried out.

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[8] If it were impossible to perform all possible definitions (14) we would be in the face of the elementary contradiction of an impossible possibility.

5. Once performed the first v definitions (14) we will have $x \leq q_v$. Therefore $x^* \leq q_v$

6. It is then impossible that $q_v < x^*$.

In consequence η cannot be an element of $\langle q_n \rangle_{n \in \mathbb{N}}$.

34 It could be claimed that x results undefined because restriction 30 is violated as a consequence of the infinitely many times x is redefined. The same type of arbitrary violation could then be claimed for any other definition of procedure composed of infinitely many successive steps, in which case infinitist mathematics would make no sense.

35 The rational number η proves, therefore, the existence of rational numbers within (a, b) which are not in $\langle q_n \rangle_{n \in \mathbb{N}}$, which in turn proves the falseness of the initial assumption on the countable nature of $\mathbb{Q}$. And taking into account Cantor’s proof on the denumerability of the set $\mathbb{Q}$, the final conclusion is that $\mathbb{Q}$ is and is not countable.

Remarck 1 The sequence of definitions (14) leads to some other contradictory results the reader could easily find. Evidently, contradictory results do not invalidate one another, they simply prove the existence of contradictions.[9] If, starting from the same hypothesis, two independent arguments lead to contradictory results they prove the inconsistency of the initial hypothesis. It is quite clear then that an argument cannot be refuted by another argument. An argument can only be refuted by indicating where and why *that* argument fails.

Remarck 2 Infinitist mathematics assumes the possibility to perform procedures of infinitely many successive steps. But mathematics is not usually concerned with the way those procedures could be, in fact, carried out; it is only concerned with the consistency of the involved arguments. When the result of a procedure or definition of infinitely many steps is an infinite set (or sequence), then the set (or sequence) is always considered as a complete infinite totality, which

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[9] This obviousness is often ignored in the discussions on the actual infinity.

implies the completion of the infinitely many steps of the corresponding procedure or definition, as in the above case of Cantor’s 1874 argument. That said, it seems appropriate to recall we dispose of a formal theory one of whose objectives is just to analyze the ways those infinite procedures and definitions could be carried out in a finite or infinite interval of time (supertask and bifurcated supertask respectively[10]).

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[10] See for instance [2], [6], [9], [7], [10], [1], [8].

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