1.-Arithmetics singularities of $\aleph_o$

INTRODUCTION

1 The discussion that follows will be concerned with the elements of the $(\omega+2)$-ordered set $\mathbb{N}^* = \{1, 2, 3, \dots, \aleph_o, 2^{\aleph_o}\}$, as well as with the basic arithmetic operations and order relations between finite and infinite cardinals introduced by Cantor in his foundational work on transfinite numbers [1]. Definitions that, essentially, continue to be applicable in modern transfinite mathematics.

2 Once assumed the existence of the set $\mathbb{N}$ of all finite cardinals (natural numbers) as a complete totality,[1] Cantor defined $\aleph_o$ as its cardinal. He then proved $\aleph_o$ is the less cardinal greater than all finite cardinals [1, Theorems 10-A and 10-B].

3 As is well known, transfinite arithmetics allows to define arithmetic operations of infinitely many operands. So, not only the operands but also the sequence of operations can be of any finite or infinite length. In the discussion that follows, and for reasons of clarity, we will index the successive operands of arithmetic operations to make it explicit the ordering of the involved operands.

[1]In modern terms: actual infinity hypothesis subsumed by the Axiom of Infinity.

Is $\aleph_o$ a prime number?

4 Axiomatic set theory,[2] that applies to all finite and infinite sets, allows us to dissociate the set $\mathbb{N}$ of natural numbers in the following way:

$$\{1, 2, 3, \dots\} = \{1\} \cup \{2, 3, 4, \dots\} \quad (1)$$
$$\{1\} \cap \{2, 3, 4, \dots\} = \emptyset \quad (2)$$

Aleph-null ($\aleph_o$) is, by definition, the cardinal[3] of $\mathbb{N}$. Taking into account the cardinal of the union of two disjoint sets is the sum of the cardinal of each set, we will have:

$$\begin{aligned} \aleph_o &= |\{1, 2, 3, \dots\}| && (3)\\ &= |\{1\} \cup \{2, 3, 4, \dots\}| && (4)\\ &= |\{1\}| + |\{2, 3, 4, \dots\}| && (5)\\ &= 1_1 + |\{2, 3, 4, \dots\}| && (6) \end{aligned}$$

where the natural number 1 is written as 1_1 to indicate it stands for the cardinal of the set $\{1\}$; the same will apply to the successive 1_2, 1_3, 1_4 etc.

5 By successive dissociations (S-dissociations from now on) of $\mathbb{N}$ we will obtain:

$$\begin{aligned} \aleph_o &= |\{1, 2, 3, \dots\}| && (7)\\ &= |\{1\} \cup \{2, 3, 4, \dots\}| && (8)\\ &= |\{1\}| + |\{2, 3, 4, \dots\}| && (9)\\ &= 1_1 + |\{2, 3, 4, \dots\}| && (10)\\ &= 1_1 + |\{2\} \cup \{3, 4, 5, \dots\}| && (11)\\ &= 1_1 + |\{2\}| + |\{3, 4, 5, \dots\}| && (12)\\ &= 1_1 + 1_2 + |\{3, 4, 5, \dots\}| && (13) \end{aligned}$$

[2]For instance ZFC-axiomatic.

[3]As is usual, the cardinal of a set X will bee denoted by $|X|$.

$$
\begin{aligned}
&= 1_1 + 1_2 + |\{3\} \cup \{4, 5, 6, \dots\}| && (14)\\
&= 1_1 + 1_2 + |\{3\}| + |\{4, 5, 6, \dots\}| && (15)\\
&= 1_1 + 1_2 + 1_3 + |\{4, 5, 6, \dots\}| && (16)\\
&\dots
\end{aligned}
$$

It is worth noting that a S-dissociation simply dissociates a set into two disjoint sets (one of them a singleton), so that the cardinal of the original set is the sum of the cardinals of the two dissociated sets.

6 The successive S-dissociations will be subjected to the following:

> Restriction 6.-*A S-dissociation will be carried out if, and only if, the result is a well defined sum of cardinals each of whose summands has an immediate predecessor, except the first one* 1_1.

7 Transfinite mathematics assumes that procedures of infinitely many steps as the above S-dissociation can in fact be carried out. On the other hand, it can easily be proved, by induction or by Modus Tollens (MT), that for each natural number v it is possible to perform the first v S-dissociations.

8 The MT proof goes as follows: Assume it is false that for every natural number v the first v S-dissociations can be carried out. If that were the case, there would exist at least a natural number n such that it is impossible to perform the first n S-dissociations. That is to say, there would exist at least a natural number n such that:

$$\aleph_o = 1_1 + 1_2 + \cdots + 1_{n-1} + |\{n, n+1, n+2, \dots\}| \tag{17}$$

cannot be S-dissociated. But this is false because:

$$
\begin{aligned}
\aleph_o &= 1_1 + 1_2 + \cdots + 1_{n-1} + |\{n, n+1, n+2, \dots\}| && (18)\\
&= 1_1 + 1_2 + \cdots + 1_{n-1} + |\{n\} \cup \{n+1, n+2, n+3, \dots\}| && (19)\\
&= 1_1 + 1_2 + \cdots + 1_{n-1} + |\{n\}| + |\{n+1, n+2, n+3, \dots\}| && (20)\\
&= 1_1 + 1_2 + \cdots + 1_{n-1} + 1_n + |\{n+1, n+2, n+3, \dots\}| && (21)
\end{aligned}
$$

Our initial assumption must therefore be false, and then we can confirm that for every natural number v the first v S-dissociations can be carried out.

9 Assume now that while the successive S-dissociations can be carried out, they are carried out. Once performed all possible S-dissociations we would have one of the following two alternatives:

$$\aleph_o = 1_1 + 1_2 + \cdots + 1_v + |\{v+1, v+2, v+3, \dots\}| \tag{22}$$

$$\aleph_o = 1_1 + 1_2 + 1_3 + \dots \tag{23}$$

where v is a certain natural number. And being v a natural number, the first alternative must be false according to 8. Consequently, once performed all possible S-dissociations we will have:

$$\aleph_o = 1_1 + 1_2 + 1_3 + \dots \tag{24}$$

10 Let us now prove (24) is an $\omega-$ordered sequence of sums. In fact, it cannot be finite because the sum of a finite number of finite summands is also finite, while $\aleph_o$ is the first infinite cardinal. So, the right side of (24) has to have an infinite number of summands. And being infinite, it can only be $\omega-$ordered, otherwise it would be at least $(\omega + 1)$-ordered[4] and then we would have:

$$\aleph_o = 1_1 + 1_2 + 1_3 + \cdots + 1_\omega + S \tag{25}$$

where S is either a sum of a finite or infinite number of the same summand 1, or 0. In any case, the summand 1_ω will always be present and it has not immediate predecessor,[5] which violates Restriction 6. In consequence (24) can only be $\omega-$ordered.

11 According to (24), and taking into account the associativity of cardinals addition and the fact that, as Cantor himself proved [1], $a^x \times a^y = a^{x+y}$ being a, x and y any three finite or infinite cardinals,

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[4] $\omega + 1$ is the less infinite ordinal greater than ω.

[5] It is the limit of all finite ordinals.

we can write:

$$2^{\aleph_o} = 2^{1_1+1_2+1_3+\dots} \quad (26)$$
$$= 2^{1_1} \times 2^{1_2+1_3+1_4+\dots} \quad (27)$$

where all $1_1, 1_2, 1_3 \dots$ stand for the first cardinal 1. Here, the subindexes simply denote the order of the corresponding summands.

12 The successive power dissociations of $2^{\aleph_o}$ (P-dissociations hereafter) would be:

$$2^{\aleph_o} = 2^{1_1+1_2+1_3+\dots} \quad (28)$$
$$= 2^{1_1} \times 2^{1_2+1_3+1_4+\dots} \quad (29)$$
$$= 2^{1_1} \times 2^{1_2} \times 2^{1_3+1_4+1_5+\dots} \quad (30)$$
$$= 2^{1_1} \times 2^{1_2} \times 2^{1_3} \times 2^{1_4+1_5+1_6+\dots} \quad (31)$$
$$\dots$$

Notice a P-dissociation is a simple application of a standard property of the product of powers.

13 The successive P-dissociations will be subjected to the following:

> Restriction 13.-*A P-dissociation will be carried out if, and only if, the result is a well defined product of powers each of whose factors has an immediate predecessor, except the first of them* 2^{1_1}.

14 Let us prove by MT that for every natural number v the first v P-dissociations can be carried out. Asume it is false that for every natural number v the first v P-dissociations can be carried out. There would exist at least a natural number n such that:

$$2^{\aleph_o} = 2^{1_1} \times 2^{1_2} \times \dots \times 2^{1_{n-1}} \times 2^{1_n+1_{n+1}+1_{n+2}+\dots} \quad (32)$$

cannot be P-dissociated. But this false because:

$$2^{\aleph_o} = 2^{1_1} \times 2^{1_2} \times \dots \times 2^{1_{n-1}} \times 2^{1_n+1_{n+1}+1_{n+2}+\dots} \quad (33)$$

$$= 2^{1_1} \times 2^{1_2} \times \cdots \times 2^{1_{n-1}} \times 2^{1_n} \times 2^{1_{n+1}+1_{n+2}+1_{n+3}+\ldots} \tag{34}$$

Therefore our initial assumption must be false and we can confirm that for every natural number v the first v P-dissociations can be carried out.

15 Assume that while the successive P-dissociations can be carried out, they are carried out. Once performed all possible P-dissociations we will have one of the following two alternatives:

$$2^{\aleph_o} = 2^{1_1} \times 2^{1_2} \times \cdots \times 2^{1_{v-1}} \times 2^{1_v+1_{v+1}+1_{n+3}+\ldots} \tag{35}$$

$$2^{\aleph_o} = 2^{1_1} \times 2^{1_2} \times 2^{1_3} \times \ldots \tag{36}$$

where v is a certain natural number. According to 14, and being v a natural number, the first alternative must be false. Consequently, once performed all possible P-dissociations we will have:

$$2^{\aleph_o} = 2^{1_1} \times 2^{1_2} \times 2^{1_3} \times \ldots \tag{37}$$

that, obviously, can also be written as:

$$2^{\aleph_o} = 2_1 \times 2_2 \times 2_3 \times \ldots \tag{38}$$

16 Let us now prove (38) is an $\omega-$ordered sequence of multiplications. In fact, it cannot be finite because the product of a finite number of finite factors is also a finite number, while $2^{\aleph_o}$ is uncountably infinite. The right side of (38) can only have an infinite number of factors. Furthermore, it must be an $\omega-$ordered sequence of factors otherwise it would be at least $(\omega+1)$-ordered and then we could write:

$$2^{\aleph_o} = 2_1 \times 2_2 \times 2_3 \times \ldots 2_\omega \times P \tag{39}$$

where P is either a product of a finite or infinite number of the same factor 2, or 1. In any case, the factor 2_ω will always be present and has not immediate predecessor,[6] which violates Restriction 13. In consequence the right side of (38) can only be $\omega-$ordered.

17 Equation (38), on the other hand, is taken for granted and, as Cantor did, can be immediately derived from Cantor's definition of cardinal exponentiation through the notion of covering [1, § 4].

18 Cantor's '*greater*' and '*less with powers* [1] will be used now to prove that $2^{\aleph_o}$ is the less transfinite cardinal that can be expressed as a product of finite cardinals greater than 1. Obviously the number of factors cannot be finite since $2^{\aleph_o}$ is uncountably infinite. Thus it has to be infinite. Let α be any transfinite ordinal of the first or of the second kind, and let $d = n_1 \times n_2 \times n_3 ...$ be the product of any α-ordered sequence of finite cardinals n_1, n_2, n_3,..., all of them greater than 1. By induction and taking into account that $2_i \leq n_i$, for every i, we would immediately prove:

$$2_1 \times 2_2 \times 2_3 \times \cdots \leq n_1 \times n_2 \times n_3 \times ... \tag{40}$$

and by transfinite induction, and taking into account that ω is the less infinite ordinal, we would prove:

$$2_1 \times 2_2 \times 2_3 \times \cdots \leq \ n_1 \times n_2 \times n_3 \times \overset{(\alpha-\text{ordered})}{\ldots} \tag{41}$$

Thus we can write:

$$2^{\aleph_o} = 2_1 \times 2_2 \times 2_3 \times \cdots \leq n_1 \times n_2 \times n_3 \times \overset{(\alpha-\text{ordered})}{\ldots} \tag{42}$$

This proves that $2^{\aleph_o}$ is the less transfinite cardinal that can be expressed as the product of an infinite sequence of finite factors greater than 1, i.e equal or greater than 2.

19 An immediate consequence of 18 is that $\aleph_o$ cannot be expressed as a product of finite cardinals greater than 1. in fact, if the number of factors is finite the product will also be finite; and if the number of factors is infinite it will be equal or greater than $2^{\aleph_o}$, which in turn

[6] It is the limit of all 2_n

is greater than $\aleph_o$. Thus, as in the case of prime numbers, $\aleph_o$ must always form part of its own factorizations.

ALEPH-NULL AND THE POWER OF THE CONTINUUM

20 Let us write the first factor 2_1 in (38) as $1_1 + 1_2$. We will have:

$$2^{\aleph_o} = (1_1 + 1_2) \times 2_2 \times 2_3 \times 2_4 \times ... \tag{43}$$

21 Taking into account the associativity of cardinal multiplications as well as the distributive property of cardinal multiplication over cardinal addition, we can successively duplicate the number of summands in the first factor of (20) by multiplying it by the successive second factors of (20):

$$\begin{aligned} 2^{\aleph_o} &= (1_1 + 1_2) \times 2_2 \times 2_3 \times 2_4 \times ... && (44)\\ &= (1_1 + 1_2 + 1_3 + 1_4) \times 2_3 \times 2_4 \times 2_5 \times ... && (45)\\ &= (1_1 + 1_2 + \cdots + 1_8) \times 2_4 \times 2_5 \times 2_6 \times ... && (46)\\ &= (1_1 + 1_2 + \cdots + 1_{16}) \times 2_5 \times 2_6 \times 2_7 \times ... && (47)\\ &= (1_1 + 1_2 + \cdots + 1_{32}) \times 2_6 \times 2_7 \times 2_8 \times ... && (48)\\ &... \end{aligned}$$

These successive duplications of the first factor of (20) will be referred to as F-duplications. They will be subjected to following:

> Restriction 21.-*An F-duplication will be carried out if, and only if, the resulting first factor is a well defined sum in which each summand 1_n has an immediate predecessor 1_{n-1}, except the first one 1_1.*

22 Let us prove, by MT, that for every natural number v the first v F-duplications can be carried out. For this, assume it is false that for every natural number v the first v F-duplications can be carried out. There would exist at least a natural number n such that it is impossible to perform the first n F-duplications. That is to say, there would exist at least a natural number n such that:

$$2^{\aleph_o} = (1_1 + 1_2 + \cdots + 1_{2^{n-1}}) \times (2_n \times 2_{n+1} \times 2_{n+2} \times \dots) \tag{49}$$

cannot be F-duplicated. It is immediate to prove this is false because:

$$2^{\aleph_o} = (1_1 + 1_2 + \cdots + 1_{2^{n-1}}) \times (2_n \times 2_{n+1} \times 2_{n+2} \times \dots) \tag{50}$$
$$= (1_1 + 1_2 + \cdots + 1_{2^{n-1}}) \times (2_n) \times (2_{n+1} \times 2_{n+2} \times 2_{n+3} \times \dots) \tag{51}$$
$$= (1_1 + 1_2 + \cdots + 1_{2^n}) \times (2_{n+1} \times 2_{n+2} \times 2_{n+3} \times \dots) \tag{52}$$

Our initial assumption must therefore be false and then we can confirm that for every natural number v the first v F-duplications can be carried out.

23 Assume now that while the successive F-duplications can be carried out, they are carried out. Once performed all possible F-duplications we would have one of the following two alternatives:

$$2^{\aleph_o} = (1_1 + 1_2 + \cdots + 1_{2^{v-1}}) \times (2_v \times 2_{v+1} \times 2_{v+2} \times \dots) \tag{53}$$
$$2^{\aleph_o} = 1_1 + 1_2 + 1_3 + \dots \tag{54}$$

where v is a certain natural number. Being v a natural number, the first alternative must be false according to 22. Consequently, once performed all possible F-duplications we will have:

$$2^{\aleph_o} = 1_1 + 1_2 + 1_3 + \dots \tag{55}$$

24 Let us now prove (55) is an $\omega-$ordered sequence of sums. In fact, it cannot be finite because the sum of a finite number of finite summands is also a finite number, while $2^{\aleph_o}$ is uncountably infinite. Therefore, the right side of (55) will have an infinite number of summands. In addition, it can only be an $\omega-$ordered sequence of summands, otherwise it would be at least $(\omega + 1)$-ordered, and then we would have:

$$2^{\aleph_o} = 1_1 + 1_2 + 1_3 + \cdots + 1_\omega + S \qquad (56)$$

where S is either a sum of finite or infinite number of the same summand 1, or 0. In any case, the summand 1_ω would always be present and has not immediate predecessor, which violates Restriction 21. In consequence the right side of (55) can only be $\omega-$ordered.

25 Taking into account (55) and (24) we can write:

$$2^{\aleph_o} = 2_1 \times 2_2 \times 2_3 \times \cdots = 1_1 + 1_2 + 1_3 + \cdots = \aleph_o \qquad (57)$$

which contradicts Cantor's theorem:

$$\aleph_o < 2^{\aleph_o} \qquad (58)$$

Remarck 1 It seems convenient to recall that argument 20/25 is exclusively based on well established definitions, operations and properties of transfinite arithmetics. It simply takes advantage of a consequence of the hypothesis of the actual infinity: the existence of $\omega-$ordered sequences as complete totalities, in spite of the fact that no last element completes them. The argument is, therefore, a formal consequence of assuming the *completion of uncompletable.* This infinitist assumption makes it possible to complete any definition or procedure composed of an $\omega-$ordered sequence of steps in which no last step completes the sequence.

Bibliography

[1] Georg Cantor, *Contributions to the founding of the theory of transfinite numbers*, Dover, New York, 1955.