1.-The relativity of simultaneity

Introduction

1 One of the best known and celebrated results of the special theory of relativity is the relativity of simultaneity: two events simultaneous in one inertial reference frame may not be simultaneous in other inertial reference frame that moves relative to the first one. They are not simultaneous if the events occur at points separated a distance greater than zero in the direction of the relative motion. The time difference between two unsynchronized clocks is often called *difference in the phase synchronization*.

2 After reviewing a score of classic and modern textbooks on the special theory of relativity one immediately comes to the conclusion that not all of their authors deduce the same difference in the phase synchronization. That difference may appear either expressed in terms of RF_o proper time or in terms of RF_v proper time.

3 Let e_1 and e_2 be two simultaneous events in their proper reference frame RF_o whose spacetime coordinates in that frame are respectively (x_{o1}, t_{o1}) and (x_{o2}, t_{o2}). Obviously we will have $t_{o1} = t_{o2}$. Let (x_{v1}, t_{v1}) and (x_{v2}, t_{v2}) be the spacetime coordinates of e_1 and e_2 in the frame RF_v from which RF_o moves with a constant velocity $0 < v < c$ in the direction from x_{o1} to x_{o2}. According to Lorentz transformation we can write:

$$t_{v1} = \left(t_{o1} + \frac{vx_{o1}}{c^2}\right)\gamma \tag{1}$$

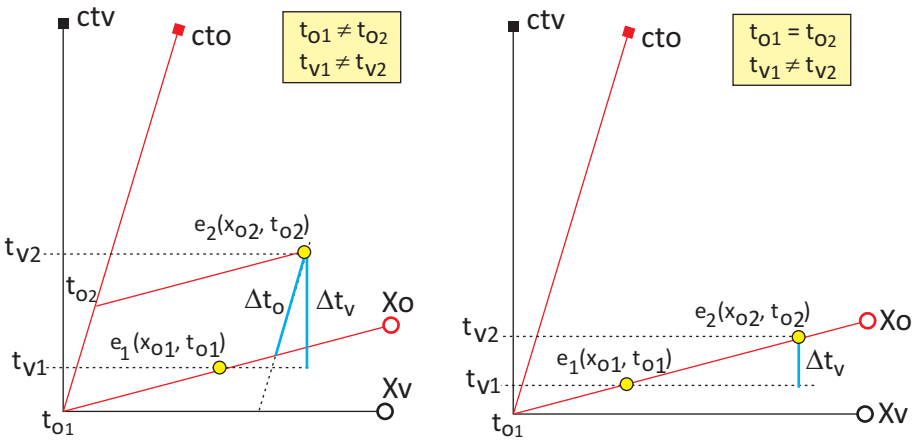

Figure 1.1: Left: The events e_1 and e_2 are not simultaneous neither in RF_o nor in RF_v. Right: Two events e_1 and e_2 that are simultaneous in their proper frame RF_o but not in RF_v.

$$t_{v2} = \left(t_{o2} + \frac{vx_{o2}}{c^2}\right)\gamma \tag{2}$$

By subtracting the first equation from the second and taking into account that $t_{o1} = t_{o2}$ we obtain:

$$t_{v2} - t_{v1} = \left(t_{o2} + \frac{vx_{o2}}{c^2}\right)\gamma - \left(t_{o1} + \frac{vx_{o1}}{c^2}\right)\gamma \tag{3}$$

$$= \left(t_{o2} - t_{o1} + \frac{vx_{o2}}{c^2} - \frac{vx_{o1}}{c^2}\right)\gamma \tag{4}$$

$$= \frac{(x_{o2} - x_{o1})v\gamma}{c^2} \tag{5}$$

$$= \frac{L_o v\gamma}{c^2} \tag{6}$$

where $L_o = x_{o2} - x_{o1}$ is the proper length of the segment $x_{o1}x_{o2}$. As noted above the time $\frac{L_o v\gamma}{c^2}$ can be contracted by a factor γ^{-1} to compensate the dilatation of time in RF_o:

$$t_{o2} - t_{o1} = \frac{vL_o}{c^2} \tag{7}$$

THE SLIDING BALLS MACHINE

4 The relativity of simultaneity is usually explained with the help of two photons that move in opposite horizontal directions in a closed space (RF_o) as a lab, a train or a spacecraft: both photons begin to move at the same instant and from the same position, the center of the lab; but while one of them move towards the left the other moves towards the right. The instants at which both photons reach their respective destinations, for instance the lab's left and right walls, are then analyzed from the perspective of two inertial reference frames: RF_o and any other inertial reference frame RF_v from which RF_o moves in a direction parallel to the trajectory of both photons. It is then proved that while in RF_o both photons reach their destinations at the same instant, in RF_v they don't.

5 In the place of photons, in the next discussion we will make use of the sliding balls machine (SBM1) depicted in Figure 1.2. SBM1 consists of two balls, $B1$ and $B2$, that slide (as the beads of an abacus) on two parallel and horizontal wires pulled by a taut inelastic rope tied to each ball at the opposite endpoints of its horizontal diameter parallel to the wire on which it is strung, as shown in the figure. Each rope is divided in a set of successive white and black marks all of them of the same proper length.

6 As Figure 1.2 also shows, an engine E makes each rope runs around three fixed pulleys and in such a way that the successive vertical white and black marks of their corresponding central sections pass always aligned, which ensures both ropes rotate at the same

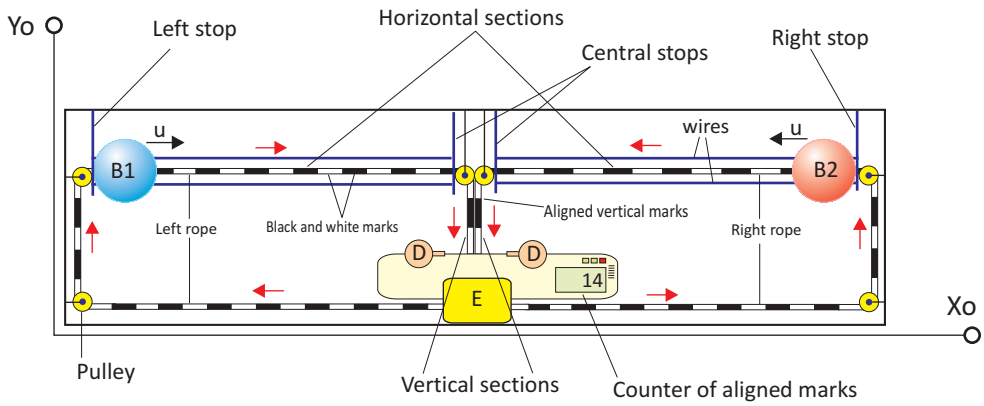

Figure 1.2: The sliding balls machine. Note the vertical marks are always aligned, which ensures both the clockwise and the anticlockwise rope rotate at the same speed.

speed, although one of them in the clockwise sense and the other in the anticlockwise sense. As a consequence of these rotations $B1$ and $B2$ slide on the wire in opposite directions and with the same speed as the (marks of the) rope.

7 Finally, a counter counts the the number of vertical alignments of the cord's white and black marks when passing over the horizontal line DD defined by the detectors D near the engine E (see Figure 1.2). This counting provides a way to calculate the speed of the balls that is independent from the relative motion of the machine.

8 The engine is also subjected to a mechanical restriction: it works only if the tension of each rope is constant along its entire length. In this sense, it could be argued that the notice of a difference in the mechanical tension in some part of the cord takes a time to reach the corresponding sensors and then the engine. Well, once elapsed that time the notice will be finally received and the engine will stop the machine.

9 We will begin by analyzing the motion of the balls from the perspective of RF_o, the proper reference frame of the machine. At instant

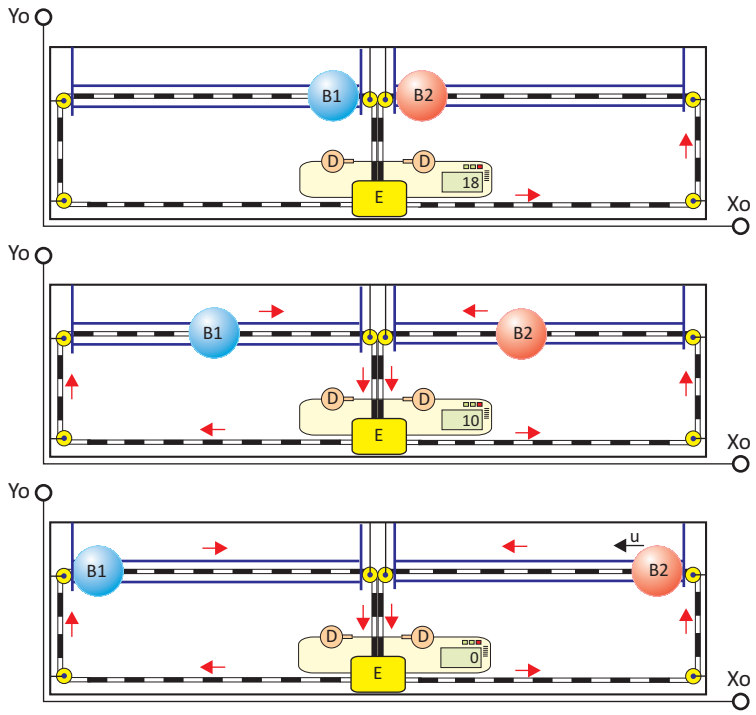

Figure 1.3: Three snapshots of the cyclic functioning of *SBM*1.

$t = 0$, the initial position of $B1$ is the left end of the machine while the initial position of $B2$ is the right one. The balls then begin to move towards the central stops with the same constant speed u. [1] Thus, and being L_o the distance each ball has to traverse, after a time $t_o = L_o/u$ the two balls reach simultaneously the central stops. The engine E then reverse its rotation making the ropes rotate in the opposite direction pulling the balls towards their initial positions with the same constant velocity u. After a time $t_o = L_o/u$ each ball recovers its initial position and a new cycle begins.

10 Let us now analyze the motion of $B1$ and $B2$ from the perspective

[1] Since they are irrelevant to our discussion, we will neglect the times elapsed in accelerations and decelerations.

of RF_v, an inertial reference frame that coincides with RF_o at instant $t = 0$, and from which RF_o moves from left to right, in the direction of X_o, with a velocity v. We are not interested here in the relative velocity of the balls with respect to RF_v but in the velocity of the balls inside the machine as is calculated from RF_v: by counting the number of marks of the vertical section. We will only analyze the motion of the balls towards the central stops.

11 In accordance with Lorentz transformation, in RF_v:

1. The horizontal section of each rope has the same length, although it is contracted by a factor γ^{-1}. Therefore $B1$ moves the same distance $\gamma^{-1}L_o$ as $B2$.
2. The vertical section of each rope has the same length as in RF_o.
3. As in RF_o, the white and black vertical marks of the central section of each rope are always seen aligned.
4. As in RF_o, $B1$ and $B2$ reach simultaneously the central ends of the machine.
5. Contrarily to RF_o, here the balls $B1$ and $B2$ don't begin to move simultaneously: $B1$ begins to move a RF_v-time $\gamma L_o v/c^2$ before $B2$ begins.

12 In RF_v (as in RF_o) the alignment of the vertical marks of the central section of each rope can only mean that both ropes move with exactly the same speed. Otherwise different parts of at least one of the ropes would move at different speeds and then its mechanical tension would not be constant along its entire length, causing the machine to halt. In consequence, in RF_v both balls $B1$ and $B2$ have also to move inside the machine with the same speed. This speed can be easily calculated from the number of vertical aligned marks counted by the counter.

13 Now then, if $B1$ and $B2$ traverses the same distance (horizontal section of each rope) at the same speed and reach their destination at the same instant, then they cannot begin to move at different instants.

14 The observers in RF_v would have to conclude the lack of simultaneity they measure in RF_o can only be apparent, as was apparent the contraction of the cord in the case of the sliding pulley (Chapter **??**) or the bending of a rod partially submerged in water.

15 Since the apparent behaviour of the machine goes against the laws of mechanics, the RF_v observers should also conclude they cannot use their distorted observations on differences in the phase synchronization to get mechanical conclusions on what happens in RF_o. By symmetry, they should also conclude their observations on length contractions, time dilatation, and mass increment with relative motion cannot be used either to get conclusions on what happens in RF_o.

16 Although RF_v observers could also conclude the lack of simultaneity is not even apparent, but a theoretical artifact that results from observing and measuring digital magnitudes by means of analog (theoretical) instruments.

17 An argument similar to the above one would apply to the reverse motion of the balls from the central stops towards their corresponding lateral stops. The conclusions on the lack of simultaneity would be the same. In this case both balls begin to move at the same instant, traverse the same distance at the same speed and, however, they don't end their respective travels at the same instant. The machine could also be rotated 180º so that $B1$ and $B2$ and their respective ropes interchange their position. The conclusions will also be the same.

18 The sliding balls machine ($SBM2$) depicted in Figure 1.4 poses some additional problems. In this machine $B1$ and $B2$ move in orthogonal directions the same distance and at the same speed as in the case of $SBM1$. In the proper frame RF_o of $SBM2$ the results are the same as in the case of $SBM1$, but in RF_v, that moves in the direction of X_o, its observers will conclude:

1. $B2$ travels a distance L_o greater than the distance $\gamma^{-1}L_o$ travelled by $B1$.

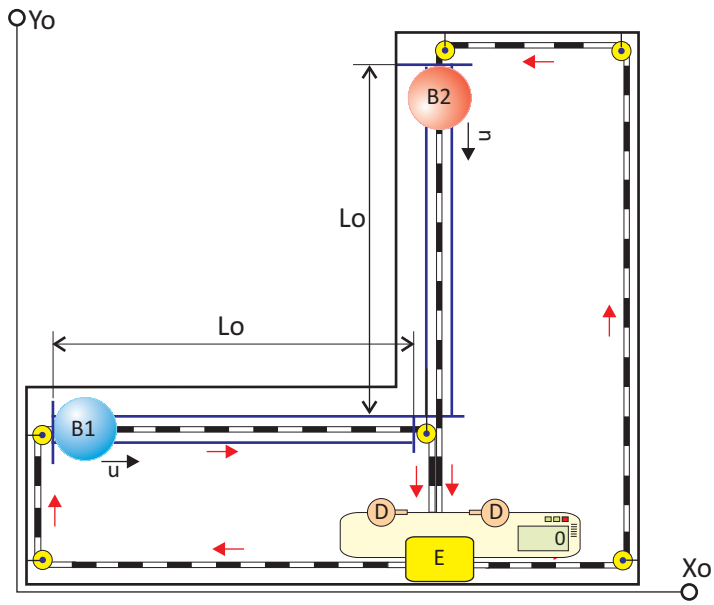

Figure 1.4: The machine *SBM*2 is similar to the machine *SBM*1, although in this case the balls *B*1 and *B*2 move in orthogonal directions.

2. *B*1 begins to move a time $\gamma v L_o/c^2$ before *B*2 begins also to move.

3. *B*1 and *B*2 move inside the machine at the same speed as is immediately deduced by counting the number of vertical alignments.

4. *B*1 and *B*2 end their travels towards the central ends of the machine at the same instant.

As in the case of *SBM*1 these conclusions are impossible: *B*2 would have to travel a distance greater than the distance travelled by *B*1 at the same speed as *B*1 and in less time than *B*1.

Bibliography