1.-Special relativity and Hooke’s law

> Even a genius as Isaac Newton was not very successful in defining inertial mass!
>
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THE CONCEPT OF MASS

1 Mass, space and time continue to be three fundamental notions of physics that, until now, we are not been able to define. They could be primitive, undefinable concepts. Although it is also possible that we have been trying to define them within an inappropriate formal framework as our current continuum-based mathematics.

2 In the case of mass we have found some operational definitions, but in the case of space and time not even that. In addition, the elementary constituents of space and time (points and instants) lack of physical meaning; they are simply considered as abstracts elements of densely ordered sets: between any tow points (or instants) infinitely many other points do exist. Thus, in the end, the special theory of relativity, which is a spacetime theory, is a theory on a physical entity on whose identity we know nothing. We will deal with spacetime in the next chapters, particularly in Chapter **??** on discrete mechanics. In this one we will discuss some relativistic issues in which mass is involved.

3 Newton referred to mass as the *quantity of matter*. Accordingly, he defined mass in terms of bulk and density [9], which evidently is a circular definition. As they have been all subsequent definitions of mass, except operational ones (5). In consequence, all axiomatic mechanics have to include mass as a primitive notion.

4 Most of the authors distinguish between gravitational and inertial mass. Even between active gravitational mass (strength of its gravitational field) and passive gravitational mass (response to a given gravitational field). Experiments clearly indicate, however, there is no physical difference between inertial and gravitational mass.

5 The inertial mass m of an object is invariably *defined* in operational terms as the ratio:

$$m = \frac{F}{a} \tag{1}$$

where F is the force applied to the object and a the resulting acceleration (Newton's second law of mechanics). But while the acceleration a can be kinematically defined in terms of most basic notions as space (x) and time (t):

$$a = \frac{d^2x}{dt^2} \tag{2}$$

force cannot.

6 To say that $m = F/a$ is not to saying much, unless we specify what a force is, without falling in a circular definition. At most, we can say that if the same force applies to two different objects O_1, and O_2 being a_1 and a_2 the resulting accelerations, then the ratio of their masses m_1/m_2 is the same as the ratio a_2/a_1 of their corresponding accelerations. If we choose, for instance, m_2 as the standard unit of mass, the inertial mass of any particle will be unambiguously determined (E. Mach operational definition).

7 As in the cases of inertial dilatation of time, phase difference in synchronization and length contraction, the inertial increment of mass, i.e. the increment of mass by relative uniform motion can also be directly derived from Lorentz transformation [7]. If m_o is the proper mass of a body B in its reference frame RF_o, the mass of that body in a reference frame RF_v that move relative to RF_o with a constant velocity v will be:

$$m_v = \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma m_o \tag{3}$$

We will discuss in the next sections some inconveniences of this relativistic increment of mass.

8 Contrarily to length contraction, time dilatation and phase difference in synchronization, that seem rather enigmatic from a pure physical perspective, the increment of mass with velocity may have a physical meaning. In effect, taking into account the existence of a limit for the maximum speed, it seems physically reasonable that the energy required to accelerate an object of mass m increases as the velocity approaches that limit

9 The assumption that mass is velocity-dependent, at least the mass of electrically charged particles, is prior to Einstein famous paper on the electrodynamics of moving bodies. Authors as Thomson [10], Heaviside [4], Abraham [1] or Lorentz [8] defended the idea that the masses of the electric particles increase with velocity [6].

10 Einstein derived his relativistic equation for mass from the equations of motion[1]:

$$m_o \frac{d^2 x_o}{dt^2} = eE_{ox} \tag{4}$$

$$m_o \frac{d^2 y_o}{dt^2} = eE_{oy} \tag{5}$$

$$m_o \frac{d^2 z_o}{dt^2} = eE_{oz} \tag{6}$$

where e is the charge of the moving particle and (E_{ox}, E_{oy}, E_{oz}) are the components of the electric field $\vec{E}_o$ in its proper inertial reference frame RF_o. Lorentz transformation and Einstein's relativistic trans-

[1]Expressed in modern notation. Einstein [2] used μ, ϵ, X' Y', Z' and β in the place, respectively, of m_o, e, E_{ox}, E_{ox}, E_{ox} and γ

formation of the electric field $\vec{E}$ and the magnetic field $\vec{B}$ lead to:

$$m_o\gamma^3\frac{d^2x_v}{dt^2} = eE_{vx} = eE_{ox} \tag{7}$$

$$m_o\gamma^2\frac{d^2y_v}{dt^2} = \gamma e[E_{vy} - (v/c)B_{vz}] = eE_{oy} \tag{8}$$

$$m_o\gamma^2\frac{d^2z_v}{dt^2} = \gamma e[E_{vz} - (v/c)B_{vy}] = eE_{oz} \tag{9}$$

which represent the motion of the same electric particle from the perspective of an inertial frame RF_v that moves relative to RF_o with a velocity v in the positive direction of the X_o axis. From both sets of equations and from Newton second law, he finally derived:

$$m_v = \gamma^3 m_o \tag{10}$$

that M. Planck converted in:

$$m_v = \gamma m_o \tag{11}$$

by replacing the definition of force used by Einstein.

11 Equation (11) can also be derived by other independent, non-relativistic argument that has nothing to do with Lorentz transformation, and that assumes the mass-energy relation[2] $E = mc^2$ as an hypothesis [3]. Indeed, assume the energy E of a body at rest is m_oc^2. According to the fundaments of classical mechanics, we can write:

$$\frac{dE}{dt} = \vec{F}\cdot\vec{v} \tag{12}$$

$$\frac{d(mc^2)}{dt} = \vec{v}\cdot\frac{d(m\vec{v})}{dt} \tag{13}$$

[2]Einstein derivation of the mass-energy relation $E = mc^2$ was considered a circular reasoning for some authors as H. E. Ives [5], which for others was an indicative of his geniality. See Chapter **??**.

$$c^2 \frac{dm}{dt} = v \frac{d(mv)}{dt} \tag{14}$$

Multiplying both sides by $2m$:

$$c^2 2m \frac{dm}{dt} = 2mv \frac{d(mv)}{dt} \tag{15}$$

and taking into account that:

$$\frac{d(c^2 m^2)}{dt} = c^2 2m \frac{dm}{dt} \tag{16}$$

$$\frac{d(mv^2)}{dt} = 2mv \frac{d(mv)}{dt} \tag{17}$$

equation (15) becomes:

$$\frac{d(c^2 m^2)}{dt} = \frac{d(m^2 v^2)}{dt} \tag{18}$$

And being equal both derivatives it must hold:

$$c^2 m^2 = m^2 v^2 + C \tag{19}$$

where C is a constant. In the special case of $v = 0$ we will have:

$$m_o c^2 = 0 + C \tag{20}$$

Replacing C with $m_o c^2$ in (19) and rearranging terms we obtain:

$$m^2 c^2 = m^2 v^2 + m_o^2 c^2 \tag{21}$$

$$m^2 = m^2 \frac{v^2}{c^2} + m_o^2 \tag{22}$$

$$m^2 (1 - \frac{v^2}{c^2}) = m_o^2 \tag{23}$$

$$m = \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}} = m_o \gamma \tag{24}$$

12 We have, therefore, two deductions for the increment of mass with velocity. The first derived from Lorentz transformation; the second from classical mechanics and the additional *hypothesis* $E = mc^2$. In the next section we will discuss some inconveniences of the first one.

Scenario

13 Consider an inertial reference frame RF_o formed by an asteroid A whose rest mass is M_o. Being inertial, A is free of any external gravitational perturbation. On the surface of A it is installed the instrument depicted in Figure 1.1 which consists of a vertical elastic band whose upper end is attached to the horizontal arm of a right angled support while in the lower one a metal sphere hangs whose rest mass is m_o. At equilibrium, the hanging mass stretches the elastic band by a length L_o. In order to avoid unnecessary discussions we will also assume the hanging mass emits a horizontal laser beam towards the vertical support activating one of the light sensors of the column of sensors placed on the vertical support. Once activated, the sensor emits a visible light whose colour is different in each sensor.

14 As pointed above (4), we will assume the equivalence between the inertial and the gravitational mass, so that a variation in one of them will be assumed to imply the same variation in the other.

15 Assuming the elastic band obeys Hooke's law and that all observers know its stretching constant k, we can write:

$$F = m_o g_a = -kL_o \tag{25}$$

where g_a is the asteroid gravity GM_o/d^2 and d the distance from the center of m_o to the center of mass of A. Taking absolute values, the observers in RF_o can write:

$$L_o = G\frac{m_o M_o}{kd^2} \tag{26}$$

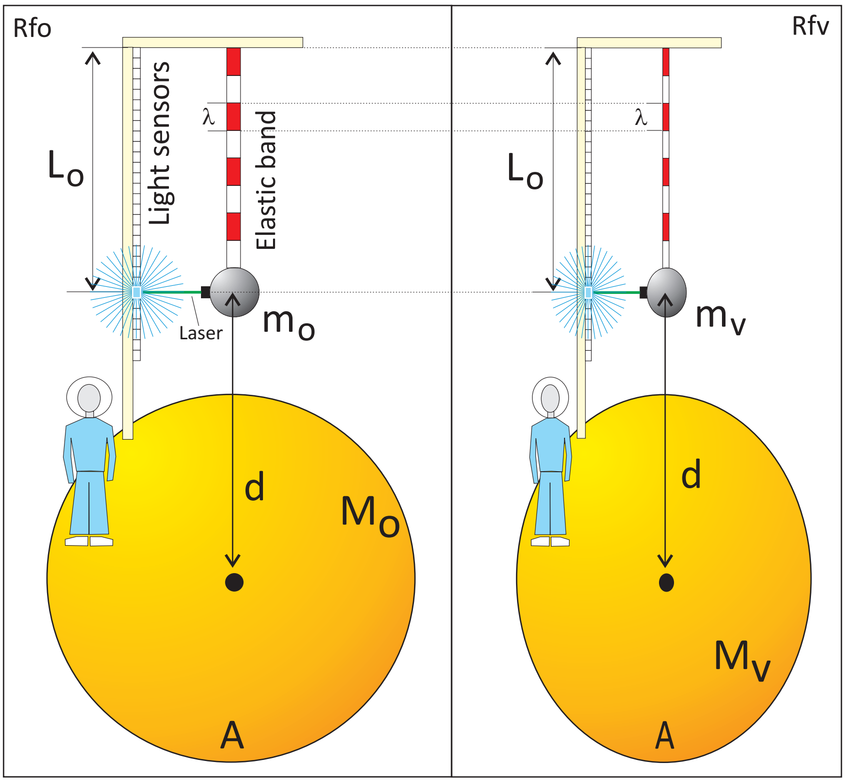

Figure 1.1: At equilibrium, and in its proper inertial frame RF_o, the weight of the hanging mass m_o stretches the elastic band by a length L_o (left). The same length L_o will be appreciate from any other inertial frame (RF_v) which moves relative to RF_o in a horizontal direction (right).

Discussion

16 Let now RF_v be any inertial reference frame from which RF_o moves with a velocity v in a direction perpendicular to the stretch direction of the elastic band. According to their relativistic *calculations*, RF_v observers can write:

$$L_v = G\frac{\gamma m_o \gamma M_o}{kd^2} = G\frac{m_o M_o}{kd^2}\gamma^2 \tag{27}$$

While, on the other hand

$$L_v = L_o \tag{28}$$

Thus, taking into account (26):

$$G\frac{m_o M_o}{kd^2} = G\frac{m_o M_o}{kd^2}\gamma^2 \tag{29}$$

which only holds if $\gamma = 1$, i.e. if $v = 0$. And, what is worse, since γ depends on the relative velocity v, the observers from different RF_v will calculate different L_v. The problem is that all of them will observe the same stretch length L_o as in RF_o.

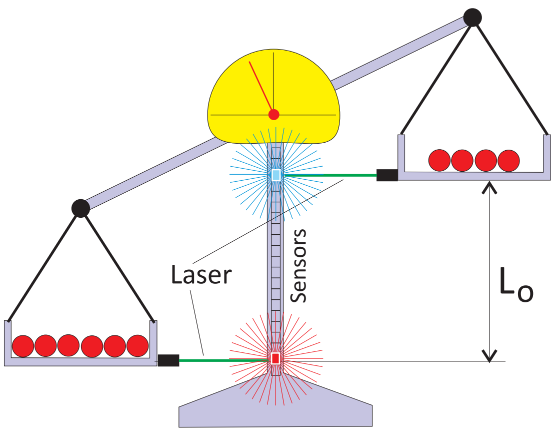

Figure 1.2: L_o is the same for all observes that move relative to the balance with a constant velocity in a direction perpendicular to its vertical support.

17 Let us now replace the elastic band on the asteroid by an equal-arm balance equipped with a system of lasers and sensors as in the elastic band case (Figure 1.2). Assume its left pan contains six balls while its right pan contains four balls, being the ten balls of identical mass m_o. In those conditions, the right pan will be raised up and the left one will be lowered, being $\Delta_o m = 2m_o$ the cause of the disequilibrium.

18 Let L_o be the vertical distance between the two panes. As in the case of the elastic band, and for exactly the same reasons, L_o will be equal for all observers at rest or in relative motion with respect to the balance in a direction perpendicular to L_o, although the cause of the disequilibrium $\Delta_v = 2\gamma m_o$ will be different for different relative velocities.

19 The conclusion is the same as in the case of the elastic band: observers from different RF_v will calculate a different L_v but all of them will observe and measure the same L_o as in RF_o if the relative motion is perpendicular to L_o.

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