

## 1.-Spacetime divisibility

### 1.1 The less infinite ordinal

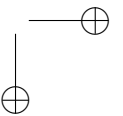
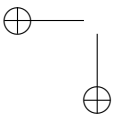
**1.1.1** The first transfinite ordinal<sup>1</sup>  $\omega$  is the less ordinal greater than all finite ordinals. It defines a type of well order called  $\omega$ -order: a set or sequence is  $\omega$ -ordered if it has a first element and every element has an immediate successor and an immediate predecessor, except the first one.<sup>2</sup> In consequence there is not a last element in an  $\omega$ -ordered set or sequence. The set  $\mathbb{N}$  of natural numbers in its natural order of precedence is a well known example of  $\omega$ -ordered set.

**1.1.2**  $\omega^*$ -Order is the symmetrical reflection of  $\omega$ -order: a set or sequence is  $\omega^*$ -ordered if it has a last element and each element has an immediate predecessor and an immediate successor, except the last one. In consequence there is not first element:

$$\underbrace{\dots t_{3^*}, t_{2^*}, t_{1^*}}_{\omega^* \text{-order}} \mid \underbrace{t_1, t_2, t_3, \dots}_{\omega \text{-order}} \quad (1)$$

<sup>1</sup>Transfinite ordinals are the ordinals of the second class according to Cantor classical terminology [4]. An ordinal of the second class is of the second kind if, as  $\omega$ , it is the limit of an infinite sequence of ordinals; it is of the first kind if it is of the form  $\alpha + n$ , where  $\alpha$  is an ordinal of the second class second kind and  $n$  a finite ordinal.

<sup>2</sup>Between an element and its immediate successor no other element of the sequence exists.



where  $1^*, 2^*, 3^*, \dots$  means last, last but one, last but two, etc.

**1.1.3** As Cantor proved,  $\omega$ -order is a formal consequence of the actual infinity hypothesis, i.e. of assuming the existence of infinite sets as complete totalities [4]. Notice that  $\omega$ -ordered sequences are completed (as the actual infinity requires) and *uncompletable* (in the sense that no last element completes them).

**1.1.4** In accordance with the definition of  $\omega$ -order given in 1.1.1, every element of an  $\omega$ -ordered set has a finite number of predecessors and an infinite number of successors. This immense asymmetry in the number of predecessors and successors ( $\omega$ -asymmetry) is a well known fact, although it is usually ignored in infinitist literature, particularly in supertask literature.

## 1.2 Super-asymmetrical supertasks

**1.2.1** At the beginning of the second half of the XX century, the discussions on the possibilities to perform a countable infinitude of actions in a finite interval of time (a supertask according to J. F. Thomson [13]) promoted the development of new infinitist theory: supertask theory.<sup>3</sup> The possibilities to perform an uncountable infinitude of actions were examined, and ruled out, by P. Clark and S. Read [5]. Supertasks have also been considered from the perspective of nonstandard analysis,<sup>4</sup> although the possibilities to perform an *hypertask* along an hyperreal interval of time have not been discussed, despite the fact that finite hyperreal intervals can be divided into hypercountably many successive infinitesimal intervals (hyperfi-

<sup>3</sup> [3], [15], [13], [14], [2]

<sup>4</sup> [11], [10], [1], [9]

nite partitions).<sup>5</sup> But most supertasks are  $\omega$ -supertasks, i.e.  $\omega$ -ordered sequences of actions performed in a finite (or perceived as finite) interval of time.

**1.2.2** Let  $\langle t_n \rangle_{n \in \mathbb{N}}$  be any  $\omega$ -ordered sequence of instants within the finite interval of time  $(t_a, t_b)$  whose limit is  $t_b$ . And let  $S$  be a supertask whose infinitely many actions  $\langle a_n \rangle_{n \in \mathbb{N}}$  are performed at the infinitely many successive instants of  $\langle t_n \rangle_{n \in \mathbb{N}}$ , each action  $a_i$  performed at the precise instant  $t_i$ .

**1.2.3** It seems convenient to recall that the limit  $t_b$  is not the instant at which  $S$  ends<sup>6</sup> but the first instant after  $S$  has ended, the first instant after performing all the infinitely many actions  $\langle a_n \rangle_{n \in \mathbb{N}}$ . Being  $t_b$  the limit of  $\langle t_n \rangle_{n \in \mathbb{N}}$ , at any instant  $t$  before  $t_b$  and arbitrarily close to it, only a finite number of tasks will have been performed and infinitely many of them remain still to be performed ( $\omega$ -asymmetry).

**1.2.4** To grasp the colossal magnitude of the above  $\omega$ -asymmetry, assume the interval  $[t_a, t_b]$  is trillions of times greater than the age of the universe and consider an interval of time  $\tau = 0.000 \dots 001$  seconds so small that we would need trillions and trillions of standard pages to write all its zeroes between the decimal point and the final digit 1, a number of pages so huge that the whole visible universe<sup>7</sup> would not have sufficient room for them; well, only a finite number of tasks will have been performed during the trillions of years elapsed between  $t_a$  and  $t_b - \tau$  while infinitely many tasks, practically all of them, will have to be performed just in our unimaginably small interval of time

<sup>5</sup>[12], [6], [8], [7], etc.

<sup>6</sup>There is not an instant at which  $S$  ends because  $\langle a_n \rangle_{n \in \mathbb{N}}$  is  $\omega$ -ordered and  $\omega$ -ordered sequences have not last element.

<sup>7</sup>A sphere of 93000 billions light years.

$\tau$ . Thus, rather than anaesthetic,  $\omega$ -asymmetry is repulsive.

### 1.3 Spacetime dichotomies

**1.3.1** Consider any finite interval of time  $[t_a, t_b]$  and within it two sequences of instants, the  $\omega$ -ordered sequence of  $t$ -instants:

$$\langle t_i \rangle : t_i = t_a + \frac{2^i - 1}{2^i} (t_b - t_a), \forall i \in \mathbb{N} \quad (2)$$

and the  $\omega^*$ -ordered sequence of  $t^*$ -instants:

$$\langle t_{i^*}^* \rangle : t_{i^*}^* = t_a + \frac{1}{2^i}, \forall i \in \mathbb{N} \quad (3)$$

where  $i^*$  stands for the last but  $(i - 1)$  element of the  $\omega^*$ -ordered sequence  $\langle t_{i^*}^* \rangle_{i \in \mathbb{N}}$ .

**1.3.2** We will examine the way the successive  $t^*$ -instants of  $\langle t_n^* \rangle_{n \in \mathbb{N}}$  and the successive  $t$ -instants of  $\langle t_n \rangle_{n \in \mathbb{N}}$  elapse as time passes from  $t_a$  to  $t_b$ , for this we will make use of the two following functions:

$$f^*(t) = \text{number of } t^* \text{-instants elapsed at } t, \forall t \in [t_a, t_b] \quad (4)$$

$$f(t) = \text{number } t \text{-instants still not elapsed at } t, \forall t \in [t_a, t_b] \quad (5)$$

**1.3.3** In accordance with the definitions of  $\omega^*$ -order and  $\omega$ -order we can write:

$$f^*(t) = \begin{cases} 0 & \text{if } t = t_a \\ \aleph_0 & \text{if } t > t_a \end{cases} \quad f(t) = \begin{cases} \aleph_0 & \text{if } t < t_b \\ 0 & \text{if } t = t_b \end{cases} \quad (6)$$

Otherwise, if there would exist an instant  $t$  such that  $f^*(t) = n$  or  $f(t) = n$ , being  $n$  a natural number, then there would also exist the

impossible firsts  $n$  elements of an  $\omega^*$ -ordered sequence or the impossible lasts  $n$  elements of an  $\omega$ -ordered sequence.

**1.3.4** According to 1.3.3, the functions  $f^*$  and  $f$  are well defined for every  $t$  in  $[t_a, t_b]$ ; they map the interval  $[t_a, t_b]$  onto the set of two elements  $\{0, \aleph_o\}$ :

$$f^* : [t_a, t_b] \mapsto \{0, \aleph_o\} \quad (7)$$

$$f : [t_a, t_b] \mapsto \{0, \aleph_o\} \quad (8)$$

**1.3.5** Function  $f^*$  defines, therefore, a dichotomy, the  $t^*$ -dichotomy:

- Regarding the number of  $t^*$ -instants elapsed when time passes from  $t_a$  to  $t_b$  only two values are possible: 0 and  $\aleph_o$ .

Function  $f$  also defines a dichotomy, the  $t$ -dichotomy:

- Regarding the number of  $t$ -instants that remain still to elapse when time passes from  $t_a$  to  $t_b$  only two values are possible:  $\aleph_o$  and 0.

**1.3.6** With respect to the number of  $t^*$ -instants elapsed from  $t_a$ , the passing of time from  $t_a$  to  $t_b$  can only exhibit two states: the state  $S^*(0)$  at which no  $t^*$ -instant has elapsed, and the state  $S^*(\aleph_o)$  at which infinitely many  $t^*$ -instants have elapsed. Intermediate finite states  $S^*(n)$  at which only a finite number  $n$  of  $t^*$ -instants have elapsed are not possible. The passing of time becomes  $S^*(\aleph_o)$  *directly* from  $S^*(0)$ . Similarly, with respect to the number of  $t$ -instants not elapsed, the passing of time from  $t_a$  to  $t_b$  can only exhibit two states:  $S(\aleph_o)$  and  $S(0)$ ; without intermediate finite states  $S(n)$  at which only a finite number  $n$  of  $t$ -instants have to elapse. The passing of time reaches the state  $S(0)$  *directly* from  $S(\aleph_o)$ .

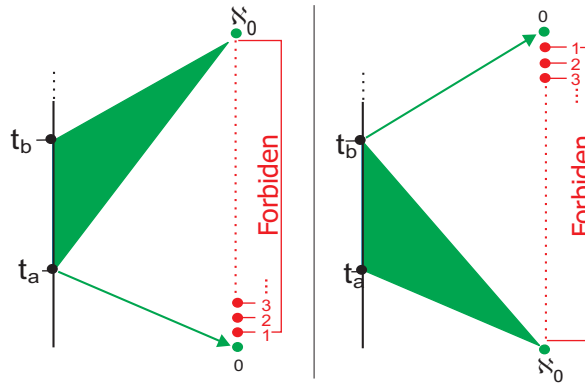


Figure 1.1:  $t^*$ -dichotomy (left) and  $t$ -dichotomy (right)

### 1.4 Spacetime divisibility

1.4.1 We will now examine the duration of the transitions:

$$S^*(0) \rightarrow S^*(\aleph_0) \tag{9}$$

$$S(\aleph_0) \rightarrow S(0) \tag{10}$$

According to (6) the number of  $t^*$ -instants elapsed from  $t_a$  and the number of  $t$ -instants still not elapsed from  $t_a$  are well defined along the whole interval  $[t_a, t_b]$ . On the other hand, both transitions must take place within the same interval  $[t_a, t_b]$ .

1.4.2 Although the real interval  $[t_a, t_b]$  is densely ordered the sequences  $\langle t_{i*}^* \rangle_{i \in \mathbb{N}}$  and  $\langle t_i \rangle_{i \in \mathbb{N}}$  contained within it, are not. These sequences are  $\omega^*$ -ordered and  $\omega$ -ordered respectively, which means that  $t^*$ -instants and  $t$ -instants are strictly successive, i.e. between any  $t^*$ -instant and its immediate successor no other  $t^*$ -instant exists; and the same applies to  $t$ -instants. Thus,  $t^*$ -instants and  $t$ -instants can only elapse successively, and in such a way that be-

tween any two of those successive instants a time greater than zero always passes. In consequence, the number of  $t^*$ -instants elapsed from  $t_a$  can only increase *one by one*, from 0 to  $\aleph_0$ . The same applies to the way the number of  $t$ -instants still not elapsed from  $t_a$  decreases from  $\aleph_0$  to 0. This successiveness will play a capital role in the subsequent discussion.

**1.4.3** As a consequence of the  $t^*$ -dichotomy, the number of  $t^*$ -instants elapsed from  $t_a$  has to increase one by one from 0 to  $\aleph_0$  without traversing the increasing sequence of natural numbers 1, 2, 3, . . . . Analogously, the number of  $t$ -instants to elapse has to decrease one by one from  $\aleph_0$  to 0 without traversing the decreasing sequence of natural numbers . . . , 3, 2, 1.

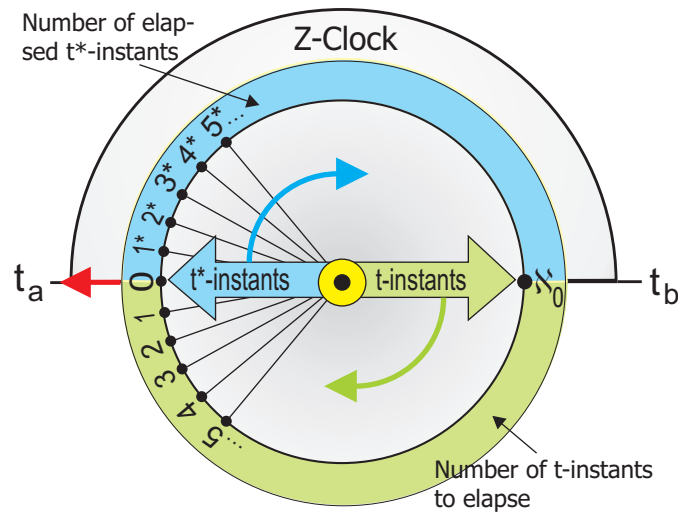


Figure 1.2: As time passes from  $t_a$  to  $t_b$  the arrow of  $t^*$ -instants will turn clockwise from 0 to  $\aleph_0$  without passing over the successive radius 1, 2, 3, . . . . At the same time the arrow of  $t$ -instants will turn clockwise from  $\aleph_0$  to 0 without passing over the successive radius . . . 3, 2, 1.

**1.4.4** The duration of the transition  $S^*(0) \rightarrow S^*(\aleph_o)$  is, according to 1.4.2, the interval of time within  $[t_a, t_b]$  during which the number of  $t^*$ -instants elapsed from  $t_a$  increases *one by one* from zero to  $\aleph_o$ . Similarly, the duration of the transition  $S(\aleph_o) \rightarrow S(0)$  is the interval of time within  $[t_a, t_b]$  during which the number of  $t$ -instants not elapsed from  $t_a$  decreases *one by one* from  $\aleph_o$  to zero.

**1.4.5** Since a time greater than zero always elapses between any two successive  $t^*$ -instants, a time greater than zero must necessarily elapse between an infinite number of successive  $t^*$ -instants. For this reason, the transition  $S^*(0) \rightarrow S^*(\aleph_o)$  will necessarily take a time greater than zero. The same conclusion applies, and for the same reason, to the transition  $S(\aleph_o) \rightarrow S(0)$ .

**1.4.6** Assume the transition  $S^*(0) \rightarrow S^*(\aleph_o)$  lasts a time  $\tau$ , being  $\tau$  any positive real number. Let  $\tau'$  be any instant within  $(0, \tau)$ . If the number of  $t^*$ -instants elapsed at  $t_a + \tau'$  were zero then the transition would not have begun at  $t_a + \tau'$  and its duration would be less than  $\tau$ ; if that number were  $\aleph_o$  the transition would have finished at  $t_a + \tau'$  and its duration would be less than  $\tau$ . But 0 and  $\aleph_o$  are the only possible values for the number of  $t^*$ -instants elapsed from  $t_a$ . In consequence the duration of  $S^*(0) \rightarrow S^*(\aleph_o)$  is less than  $\tau$ . And being  $\tau$  any real number greater than 0, we must conclude the duration of  $S^*(0) \rightarrow S^*(\aleph_o)$  is less than any real number greater than zero. Or in other words, it lasts a null time.

**1.4.7** An argument similar to 1.4.6 proves the transition  $S(\aleph_o) \rightarrow S(0)$  has also to be instantaneous. It could be argued that the transition  $S(\aleph_o) \rightarrow S(0)$  lasts a time  $t_b - t_a$ , but this is impossible because at  $t_a + \tau$ , being  $\tau$  any instant within  $(0, t_b - t_a)$ , the number of  $t$ -instants

still not elapsed is  $\aleph_o$ , and then the transition  $S(\aleph_o) \rightarrow S(0)$  has not begun. In consequence it last an amount of time less than  $t_b - t_a$ .

**1.4.8** According to 1.4.6 and 1.4.7, infinitely many successive  $t^*$ -instants and infinitely many successive  $t$ -instants have to elapse simultaneously. But this is impossible because successive instants cannot elapse simultaneously: between any two of those successive instants  $t_n^*, t_{n+1}^*$  (or  $t_n, t_{n+1}$ ) a finite interval of time greater than zero always passes: just the interval  $[t_n^*, t_{n+1}^*]$  (or the interval  $[t_n, t_{n+1}]$  in the case of  $t$ -instants). The transitions  $S^*(0) \rightarrow S^*(\aleph_o)$  and  $S(\aleph_o) \rightarrow S(0)$  have to be greater than zero, but they cannot be greater than zero (1.4.6/1.4.7). We have therefore two contradictions proving the impossibility of dividing any finite interval of time into an actual infinitude of  $\omega^*$ -ordered parts and into an actual infinitude of  $\omega$ -ordered parts (see Z-Clock in Figure 1.2).

**1.4.9** Any infinite partition of time has to be  $\alpha$ -ordered, being  $\alpha$  an ordinal of the second class (first or second kind). Thus, we will have:

$$\alpha = \omega \tag{11}$$

$$\text{or} \tag{12}$$

$$\alpha = \omega + \beta \tag{13}$$

where  $\beta$  is an ordinal of the second class (first or second kind). In consequence, any transfinite partition of time has to contain at least an impossible  $\omega$ -ordered partition. Time is not, therefore, infinitely divisible.

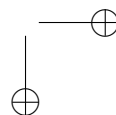
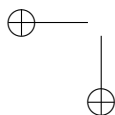
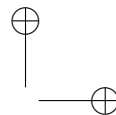
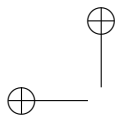
**1.4.10** If in the place of the passage of time and the sequences of  $t^*$ -instants and  $t$ -instants we would have considered the uniform

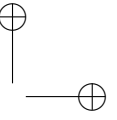
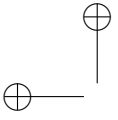
linear motion of a particle traversing the  $Z^*$  points  $\langle z_n^* \rangle_{n \in \mathbb{N}}$  and  $Z$ -points  $\langle z_n \rangle_{n \in \mathbb{N}}$  defined within the real interval  $[0, 1]$  of the real line as:

$$\langle Z_{j^*}^* \rangle : z_{j^*}^* = \frac{1}{2^j}, \forall j \in \mathbb{N} \quad (14)$$

$$\langle Z_i \rangle : z_i = \frac{2^i - 1}{2^i}, \forall i \in \mathbb{N} \quad (15)$$

We would have come to the same conclusion on the infinite divisibility of space we have come on the infinite divisibility of time.

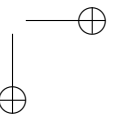
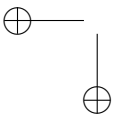




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