1.-Relativistic refraction of light

THE REFRACTIVE INDEX

1 Maxwell equations lead almost immediately to the the wave equation for electric fields:

$$\nabla^2 \overrightarrow{E} = \mu_o \epsilon_o \frac{\partial^2 \overrightarrow{E}}{\partial^2 t} \tag{1}$$

and for magnetic fields:

$$\nabla^2 \overrightarrow{B} = \mu_o \epsilon_o \frac{\partial^2 \overrightarrow{B}}{\partial^2 t} \tag{2}$$

being μ_o and ϵ_o the magnetic permeability[1] and the electric permittivity[2] respectively of the free space (vacuum), two universal constants as Planck constant, or the universal constant of gravitation. Compared with the standard form of a wave equation:

$$\nabla^2 \overrightarrow{Y} = \frac{1}{v^2} \frac{\partial^2 \overrightarrow{Y}}{\partial^2 t} \tag{3}$$

[1]A mesure of the magnetization of a medium in response to a magnetic field.

[2]A measure of the electric distortion of a medium in response to an electric field.

one immediately infers, as Maxwell did, that:

$$v = \frac{1}{\sqrt{\mu_o \epsilon_o}} \tag{4}$$

$$= \frac{1}{\sqrt{4\pi \times 10^{-7} mKgC^{-2} \times 8.8541878 \times 10^{-12} C^2 s^2 Kg^{-1} m^{-3}}} \tag{5}$$

$$= 299790 Km/s \tag{6}$$

is the speed of the electromagnetic waves, and then the speed of light in the vacuum, usually denoted by c. Evidently, c is also a universal constant.

2 Each material medium M has its own magnetic permeability μ_m and its own electric permittivity ϵ_m, usually greater than those of the vacuum. Light then travels through a transparent medium M with a speed v less than c given by:

$$v = \frac{1}{\sqrt{\mu_m \epsilon_m}} \tag{7}$$

3 The relative permeability $\mu_r = \mu_m / \mu_o$ and the relative permittivity $\epsilon_r = \epsilon_m / \epsilon_o$ are frequently used in the place of μ_m and ϵ_m. They represent the extent to which the material's permeability and permittivity exceed those of free space. These relative magnitudes allow us to write:

$$v = \frac{1}{\sqrt{\mu_m \epsilon_m}} = \frac{1}{\sqrt{\mu_r \mu_o \epsilon_r \epsilon_o}} \tag{8}$$

$$= \frac{1}{\sqrt{\mu_r \epsilon_r}} \frac{1}{\sqrt{\mu_o \epsilon_o}} \tag{9}$$

$$= \frac{c}{\sqrt{\mu_r \epsilon_r}} \tag{10}$$

$$= \frac{c}{n} \tag{11}$$

where $n = \sqrt{\mu_r \epsilon_r} = c/v$ is the refractive index of M. As noted in Chapter **??** on conventions, we will always use light of the same frequency.

4 With respect to its optical properties, ordinary matter can be isotropic or anisotropic. Crystalline solids, except those the isometric (cubic) system of symmetry, are anisotropic for the index of refraction: the index varies with direction. In the case of isotropic substances the index of refraction does not change with direction.

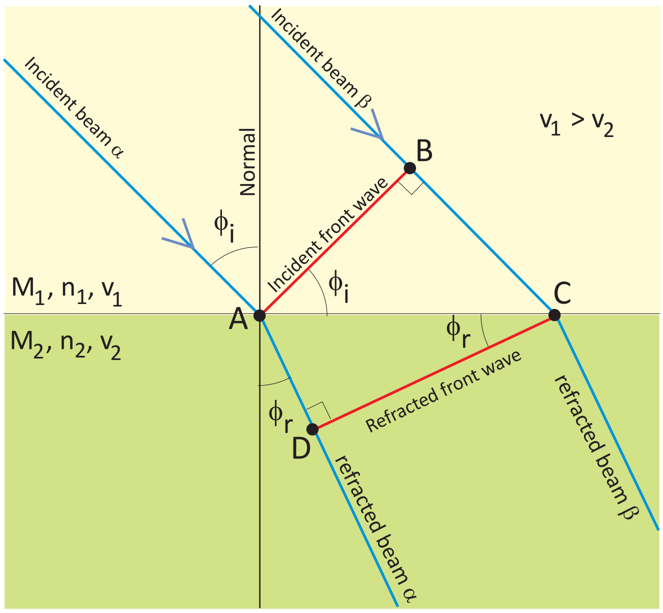

Figure 1.1: During a time t the incident beam β travels from B to C at a velocity v_1 while the refracted beam α travels from A to D at a velocity v_2. This fact and some elementary trigonometry suffices to derive Snell's law.

5 As light crosses from one medium (the vacuum for instance) into another its velocity changes. As a consequence of that change the wave front deviates its trajectory and the light beams bend in the frontier between both media. This phenomenon is the familiar refraction of light, the reason for which a rod partially and obliquely submerged in water seems to be bent just at the interface between air and water.

6 The refraction of light follows Snell's law, a simple arithmetic relation between the incidence and refraction angles and the refractive indexes of the media between which the refraction occurs. To derive Snell's law, consider two media M_1 and M_2 whose respective indexes of refraction are n_1 and n_2, being therefore $v_1 = c/n_1$ and $v_2 = c/n_2$ the speed of light in M_1 and M_2 respectively. Let us now consider the two rays of light α and β depicted in Figure 1.1. The distance travelled by the refracted beam α at a velocity v_2 and the distance travelled by the incident beam β at the velocity v_1 during a given time t allow us to write:

$$t = \frac{AD}{v_2} = \frac{BC}{v_1} \tag{12}$$

Thus:

$$\frac{v_1}{v_2} = \frac{BC}{AD} \tag{13}$$

And being:

$$\sin\phi_i = \frac{BC}{AC};\ \sin\phi_r = \frac{AD}{AC} \tag{14}$$

we can write:

$$\frac{\sin\phi_i}{\sin\phi_r} = \frac{BC}{AD} \tag{15}$$

$$= \frac{v_1}{v_2} \tag{16}$$

And taking into account (11) ($v = c/n$):

$$\frac{\sin\phi_i}{\sin\phi_r} = \frac{\frac{c}{n_1}}{\frac{c}{n_2}} \tag{17}$$

$$= \frac{n_2}{n_1} \tag{18}$$

which is the arithmetic expression for Snell's law. If M_1 is the vacuum then $n_1 = 1$ and we have another expression for the index of refraction of M_2:

$$n_2 = \frac{\sin \phi_i}{\sin \phi_r} \tag{19}$$

Relativity of light refraction

7 Contrarily to what happens with distances, the electromagnetic permittivity and permeability of the vacuum are independent of relative motion, both are universal constants. The same is assumed of the electromagnetic permittivity and permeability of material media. Thus, although the speed of light in a particular isotropic medium is different from the speed of light in the vacuum, it is also considered as a universal constant that does not depend upon relative motion.

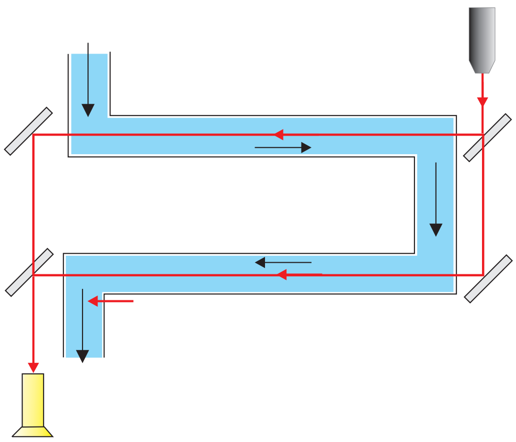

Figure 1.2: Schematic representation of Fizeau experiment: light propagates through a moving medium, first in opposite direction and then in the same direction as the flowing water

8 Therefore, the index of refraction is also independent of relative motion. Fizeau experiment is an empirical proof of this conclusion. Indeed, Fizeau drag coefficient results from the relativistic addition of velocities: the velocity of light in water (c/n, being n the refractive index of water) and the velocity v of the water inside the pipe. For the case in which light and water move in the same direction, light will move with a velocity c_1 that results from the relativistic addition of c/n and v:

$$c_1 = \frac{c/n + v}{1 + \frac{c/n}{c}\frac{v}{c}} \tag{20}$$

$$= \frac{c}{n}\left(1 + \frac{vn}{c}\right)\left(1 + \frac{v}{nc}\right)^{-1} \tag{21}$$

$$= \frac{c}{n}\left(1 + \frac{vn}{c}\right)\left(1 - \frac{v}{nc} + \ldots\right) \tag{22}$$

$$\approx \frac{c}{n}\left(1 + \frac{vn}{c} - \frac{v}{nc}\right) \tag{23}$$

$$= \frac{c}{n} + v\left(1 - \frac{1}{n^2}\right) \tag{24}$$

where $(1 - 1/n^2)$ coincides with Fizeau drag coefficient, and n is used as a universal constant. Some modern variants of Michelson-Morley experiment make also use of different transparent media with different indexes of refraction, and in all cases the speed of light in each of those media is considered as a universal constant independent of relative motion.

9 The variation of the refractive index with relative motion would mean the variation with relative motion of the speed of light through optically isotropic media. A variation that is not considered by Lorentz transformation and then one that could not be explained within the theoretical framework of the special theory of relativity. In addition that variation would have unacceptable implications on the observance of the laws of optical crystallography. In consequence, in the remainder of this book we will assume the index of refraction does not change with relative motion.

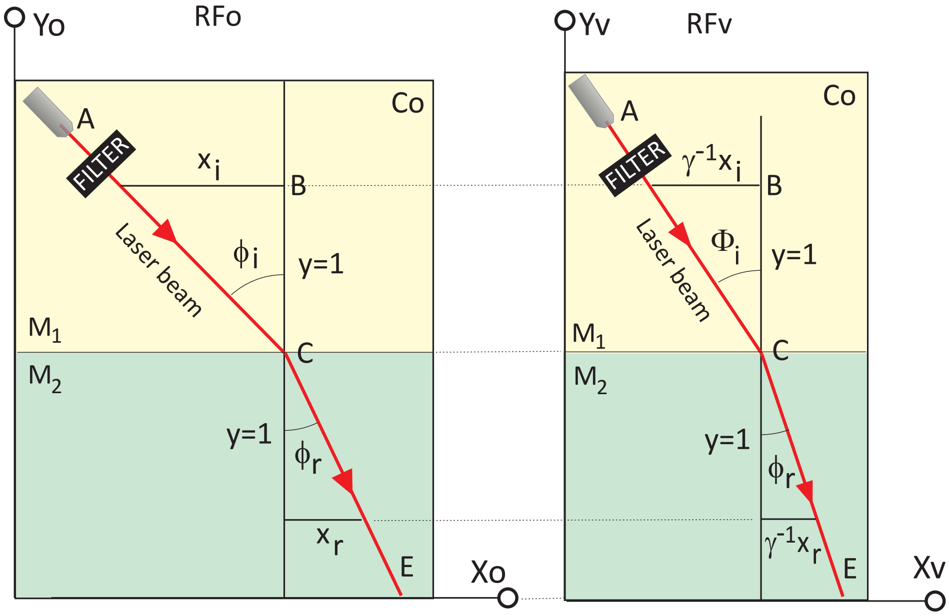

Figure 1.3: Refraction of a visible laser beam in the proper frame RF_o of its emitting source (left). The same refraction as seen from RF_v that moves relative to RF_o (right).

10 We will examine now a relativistic conflict involving Snell's law. Let M_1 (with refractive index n_1) and M_2 (with refractive index n_2) be two transparent media in a container Co in the frame RF_o, as Figure 1.3 shows. According to Snell's law we can write:

$$\frac{n_2}{n_1} = \frac{\sin \phi_i}{\sin \phi_r} \tag{25}$$

for any pair of incident and refracted rays. Consider the incident visible[3] laser beam AC and its corresponding refracted beam CE depicted in Figure 1.3. To ensure all observers agree in the frequency of the laser beam, it will traverse a filter from which only the waves of a predetermined wave length and frequency emerge. In these conditions the disagreement would imply the malfunctioning of all filters

[3]See Chapter **??** on conventions

of light frequency. Equation (25) can be written as:

$$\frac{n_1}{n_2} = \frac{\dfrac{x_i}{(x_i^2+1)^{1/2}}}{\dfrac{x_r}{(x_r^2+1)^{1/2}}} \tag{26}$$

$$= \frac{x_i}{x_r} \times \frac{(x_r^2+1)^{1/2}}{(x_i^2+1)^{1/2}} \tag{27}$$

where the legs x_i and x_r are parallel to the X_0 axis of RF_o, the proper frame of M_1 and M_2.

11 From the point of view of the frame RF_v, that moves relative to RF_o with a velocity v in the X_o direction, equation (27) becomes:

$$\frac{n_1}{n_2} = \frac{\gamma^{-1}x_i}{\gamma^{-1}x_r} \times \frac{(\gamma^{-2}x_r^2+1)^{1/2}}{(\gamma^{-2}x_i^2+1)^{1/2}} \tag{28}$$

$$= \frac{x_i}{x_r} \times \frac{(\gamma^{-2}x_r^2+1)^{1/2}}{(\gamma^{-2}x_i^2+1)^{1/2}} \tag{29}$$

and taking into account (27) we obtain:

$$\frac{x_i}{x_r} \times \frac{(x_r^2+1)^{1/2}}{(x_i^2+1)^{1/2}} = \frac{x_i}{x_r} \times \frac{(\gamma^{-2}x_r^2+1)^{1/2}}{(\gamma^{-2}x_i^2+1)^{1/2}} \tag{30}$$

Therefore:

$$\frac{x_r^2+1}{x_i^2+1} = \frac{\gamma^{-2}x_r^2+1}{\gamma^{-2}x_i^2+1} \tag{31}$$

which only holds if $x_i = x_r$ (and then $n_1 = n_2$), or if $\gamma = 1$ (in whose case we will have $v = 0$). Indeed:

$$(x_r^2+1)(\gamma^{-2}x_i^2+1) = (x_i^2+1)(\gamma^{-2}x_r^2+1) \tag{32}$$

$$\gamma^{-2}x_i^2x_r^2 + x_r^2 + \gamma^{-2}x_i^2 + 1 = \gamma^{-2}x_i^2x_r^2 + x_i^2 + \gamma^{-2}x_r^2 + 1 \tag{33}$$

$$x_r^2 + \gamma^{-2} x_i^2 = x_i^2 + \gamma^{-2} x_r^2 \tag{34}$$

$$x_r^2 - \gamma^{-2} x_r^2 = x_i^2 - \gamma^{-2} x_i^2 \tag{35}$$

$$x_r^2 (1 - \gamma^{-2}) = x_i^2 (1 - \gamma^{-2}) \tag{36}$$

which only holds if $\gamma^{-2} = 1$ and then $v = 0$, or $x_i = x_r$ and then no refraction occurs. In consequence, while Snell's law holds for any proper refraction in RF_o, it does not hold when observed in relative motion, as in the case of RF_v.

TOTAL REFLECTION OF LIGHT

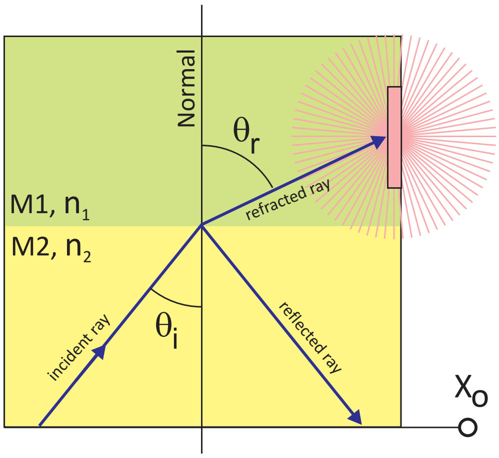

Figure 1.4: Reflection and refraction of a light beam crossing from a medium $M2$ whose refractive index is n_2 to another $M1$ whose index of refraction is $n_1 < n_2$. The refracted beam activates a sensor that emit a visible red flash making it evident the refraction is taking place.

12 We will examine now the case of the total internal reflection of light as seen from two inertial reference frames that move relative to each other. Let then RF_o be the proper frame of $M1$ and $M2$, two isotropic transparent media whose refractive indexes are n_1 and n_2 respectively, being $n_1 < n_2$. We know that as light crosses from a medium with an index of refraction n_2 into another with a less index of refraction n_1, ($n_1 < n_2$) it is partially reflected and partially refracted. As the angle of incidence α increases, the amount of refracted light decreases. There is a critical angle α_{oc} above which no refraction occurs and all incident light is reflected. That critical angle is related to the indexes of refraction by:

$$sin\,\alpha_{oc} = \frac{n_1}{n_2} \tag{37}$$

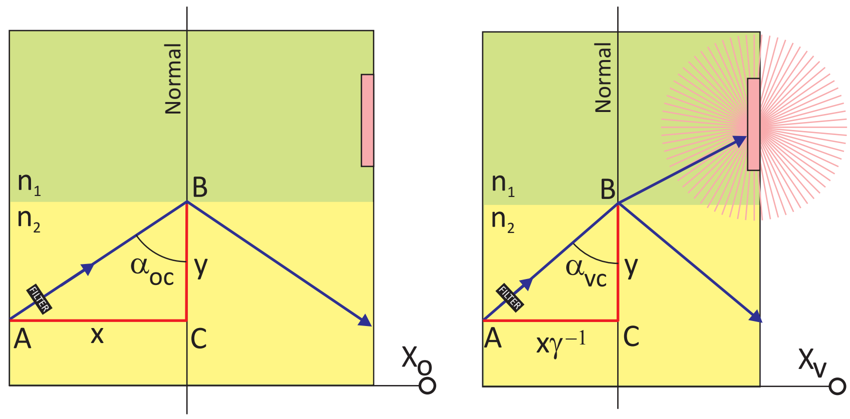

Figure 1.5: According to FitzGerald-Lorentz contraction the total reflection of light may, or not, occur depending on the relative motion at which it is observed. As a consequence, the critical angle, which depends only on the refractive indexes, not only depends on the refractive indexes.

13 Let us now examine the critical angle α_{vc} from the point of view of an inertial frame RF_v that moves relative to RF_o with at a velocity v parallel to the X_o axis of RF_o. As we have just seen, in RF_o the critical angle α_{oc} is given by (37). To calculate the critical angle α_{oc} in RFo, consider the right angled triangle ABC in Figure 1.5. We can write:

$$\alpha_{oc} = tan^{-1}\frac{x}{y} \tag{38}$$

where x is parallel to X_o. From the point of view of RF_v the leg y is the same as in RF_o but x is contracted by a factor γ^{-1}. In consequence, the observers of RF_v measure:

$$\alpha_{vc} = tan^{-1}\frac{\gamma^{-1}x}{y} \tag{39}$$

and being $\gamma^{-1}x < x$, we can write:

$$\alpha_{vc} \; < \; \alpha_{oc} \tag{40}$$

On the other hand, and taking into account that n_1/n_2 does not change with relative motion, we can also write:

$$\alpha_{vc} = \frac{n_1}{n_2} = \alpha_{oc} \tag{41}$$

We have therefore:

$$\alpha_{vc} < \alpha_{vc} \;\wedge\; \alpha_{vc} = \alpha_{vc} \tag{42}$$

which is, certainly, contradictory.

14 Observers from RF_v could therefore conclude that FitzGerald-Lorentz contraction can only be apparent and should not be used to get conclusions on certain physical phenomena that takes place in other reference frames, in the same way that the bending of the rod partially submerged in water should not be use to get conclusions on the physical efforts that have been applied or are being applied to the rod.

Bibliography