1.-On the derivation of the mass-energy relation

1.1 Introduction

1.1.1 As is well known, $E = mc^2$ is the most popular of all scientific formulas found by man. And not just popular, its scientific and human consequences have been enormous since its discovering. A discovering that is universally attributed to Albert Einstein. Not so popular are, however, the academic discussions on the true authorship of the discovering.

1.1.2 Since we are not interested here in that type of discussion, we shall limit ourselves to enumerate some of the less controversial facts related to the authorship of the discovering of the mass-energy relation:

1. In the penultimate query of his Opticks (Query 30) [6], Newton asked: Are not the gross bodies and light convertible into one another, and may not bodies receive much of their activity from the particles of light which enter their composition? The changing of light into bodies and bodies into light is very conformable to the course of nature.

2. The concept of electromagnetic mass was developed along the

XIX century by authors as J. J. Thomson, O. Heaviside or H. Lorentz. By this concept they tried to understand the way the electromagnetic field contributes to the mass of charged particles.

3. In 1900 H. Poincaré published a paper [8] in which he derived the expression $M = S/c^2$, where M is the momentum of electromagnetic radiation, S de flux of radiation and c the speed of light.

4. In a new paper [9], this time published in 1904, H. Poincaré formulated his Principle of Relativity that states: It is impossible by observation made on a body to detect its uniform motion or translation. This principle together with the above relation $M = S/c^2$ lead to $\Delta m = E/c^2$.

5. In 1904 and 1905, just before the publication of Einstein's paper on the electrodynamic of moving bodies [2], and in the same journal Annalen der Phyisik, Friedrich Hasenhörl published two papers on the theory of radiation in moving bodies [3], [4] stating for the first time the explicit declaration that the heat energy of a body increments its mass.

6. In November 1905, Einstein published a short paper [1] in which he derived his famous mass-energy relation by making use of a result proved in his previous paper on the electrodynamic of moving objects. In this paper he made no reference to previous works on this issue.

7. M. Planck published a paper in 1907 [7] deriving the same energy-mass relation as Einstein, but from Poincaré's momen-

tum of radiation. In this paper Planck acknowledged the priority of Einstein's work, although judging his own approach as more general than Einstein's one.

8. In a short paper published in 1952 [5], H. E. Ives reviewed the contribution of some authors to the derivation of the mass energy relation. In a final appendix he proved the circularity of Einstein argument.

1.1.3 In the next two sections we will reproduce both Einstein's derivation on the mass-energy relation and Ives' proof on its circularity

1.2 Einstein's derivation of the mass-energy relation

1.2.1 This section reproduces a translation by W. Perrett and G.B. Jeffery (in the public domain) of Einstein's original work: Ist die Trägheit eines Körpers von seinem Energieinhalt abhänging? (Does the inertia of a body depend upon its energy-content?). For the sake of simplicity and clarity, Einstein original notation will be partially changed, in particular V, that in Einstein's notation stands for the speed of light, will replaced by c; and γ will be used to denote $1/\sqrt{(1-v^2/c^2)}$.

> The results of the previous investigation [2] lead to a very interesting conclusion, which is here to be deduced. I based that investigation on the Maxwell-Hertz equations for empty space, together with the Maxwellian expression for the electromagnetic energy of space, and in addition the principle that:

> The laws by which the states of physical systems alter are independent of the alternative, to which of two systems of coordinates, in uniform motion of parallel translation relatively to each other, these alterations of state are referred (principle of relativity).

With these principles[1] as my basis I deduced inter alia the following result (§8):

Let a system of plane waves of light, referred to the system of co-ordinates (x, y, z), possess the energy ℓ; let the direction of the ray (the wave-normal) make an angle φ with the X axis of the system. If we introduce a new system of coordinates (ξ, η, ζ) moving in uniform parallel translation with respect to the system (x, y, z), and having its origin of coordinates in motion along the X axis with the velocity v, then this quantity of lightŮmeasured in the system possesses the energy ℓ^*:

$$\ell^* = \gamma\ell\left(1 - \frac{v}{c}\right)\cos\varphi \tag{1}$$

where c denotes the velocity of light. We shall make use of this result in what follows.

Let there be a stationary body in the system (x, y, z), and let its energy referred to the system (x, y, z) be E_o. Let the energy of the body relative to the system (ξ, η, ζ) moving as above with the velocity v, be H_o.

[1]The principle of the constancy of the velocity of light is of course contained in Maxwell’s equations.

Let this body send out, in a direction making an angle φ with the X axis, plane waves of light of energy $L/2$ measured relatively to (x, y, z), and simultaneously an equal quantity of light in the opposite direction. Meanwhile the body remains at rest with respect to the system (x, y, z). The principle of energy must apply to this process, and in fact (by the principle of relativity) with respect to both systems of coordinates. If we call the energy of the body after the emission of light E_1 and H_1 respectively, measured relatively to the system (x, y, z) and (ξ, η, ζ) respectively, then by employing the relation given above we obtain:

$$E_o = E_1 + \left[\frac{L}{2} + \frac{L}{2}\right] \tag{2}$$

$$H_o = H_1 + \left[\gamma\frac{L}{2}\left(1 - \frac{v}{c}\right)\cos\varphi + \gamma\frac{L}{2}\left(1 + \frac{v}{c}\right)\cos\varphi\right] \tag{3}$$

$$= H_1 + \gamma L \tag{4}$$

By subtraction we obtain from these equations:

$$(H_o - E_o) - (H_1 - E_1) = L(\gamma - 1) \tag{5}$$

The two differences of the form $H - E$ occurring in this expression have simple physical significations. H and E are energy values of the same body referred to two systems of coordinates which are in motion relatively to each other, the body being at rest in one of the two systems (system (x, y, z)). Thus it is clear that the difference $H - E$ can differ from the kinetic energy K of the body, with respect to the other system (ξ, η, ζ), only by an additive constant C, which depends on the choice of the arbitrary additive constants of the energies H and E. Thus we may place

$$H_o - E_o = K_o + C \quad (6)$$

$$H_1 - E_1 = K_o + C \quad (7)$$

since C does not change during the emission of light. So we have

$$K_o - K_1 = L(\gamma - 1) \quad (8)$$

The kinetic energy of the body with respect to (ξ, η, ζ) diminishes as a result of the emission of light, and the amount of diminution is independent of the properties of the body. Moreover, the difference $K_o - K_1$, like the kinetic energy of the electron (§10), depends on the velocity.

Neglecting magnitudes of fourth and higher orders we may place:

$$K_o - K_1 = \frac{L}{c^2}\frac{v^2}{2} \quad (9)$$

From this equation it directly follows that:

If a body gives off the energy L in the form of radiation, its mass diminishes by L/c^2. The fact that the energy withdrawn from the body becomes energy of radiation evidently makes no difference, so that we are led to the more general conclusion that:

> The mass of a body is a measure of its energy-content; if the energy changes by L, the mass

> changes in the same sense by $L/(9 \times 10^{20}$, the energy being measured in ergs, and the mass in grammes.

> It is not impossible that with bodies whose energy-content is variable to a high degree (e.g. with radium salts) the theory may be successfully put to the test.
>
> If the theory corresponds to the facts, radiation conveys inertia between the emitting and absorbing bodies.

1.2.2 Notice the popular formula $E = mc^2$ does not explicitly appears in Einstein's paper, although it can be immediately derived from the last of its equations:

$$K_o - K_1 = \frac{L}{c^2}\frac{v^2}{2} \tag{10}$$

Indeed, on the one hand, $K_o - K_1$ is greater than zero. On the other, and being constant the velocity v of the body, its kinetic energy can only change by a change of mass. Finally, since the only event is the emission of electromagnetic radiation by the body, only this emission can account for the change in the body's mass. So, we have:

$$K_o - K_1 = \frac{1}{2}\Delta m v^2 \tag{11}$$

where Δm is the change of mass due to the emission of electromagnetic radiation. We can write:

$$\frac{1}{2}\Delta m v^2 = \frac{1}{2}L\frac{v^2}{c^2} \tag{12}$$

And then:

$$\Delta m = L/c^2 \tag{13}$$

Or:

$$L = \Delta mc^2 \tag{14}$$

1.3 Ives' criticism

1.3.1 Before developing Ives' argument on the circularity of the above Einstein's derivation of the mass-energy relation, we must recall the relativistic expression for the kinetic energy K, which is the work done by a net force F acting on a particle, being in turn the force equal to the change of the particle's relativistic momentum $p = \gamma mv$:

$$K = \int_0^{v_f} Fds = \int_0^{v_f} \frac{dp}{dt} ds \tag{15}$$

$$= \int_0^{v_f} vdp = \int_0^{v_f} vd\left(\frac{mv}{\sqrt{1-v^2/c^2}}\right) \tag{16}$$

where the last differential may be resolved as:

$$d\left(\frac{mv}{\sqrt{1-v^2/c^2}}\right) = \left(m\left(1-\frac{v^2}{c^2}\right)^{-1/2} + m\frac{v^2}{c^2}\left(1-\frac{v^2}{c^2}\right)^{-3/2}\right)dv \tag{17}$$

$$= m\left(1-\frac{v^2}{c^2}\right)^{-3/2}\left(\left(1-\frac{v^2}{c^2}\right)+\frac{v^2}{c^2}\right)dv \tag{18}$$

$$= m\left(1-\frac{v^2}{c^2}\right)^{-3/2} dv \tag{19}$$

And then we can rewrite (16) as:

$$K = \int_0^{v_f} vd\left(\frac{mv}{\sqrt{1-v^2/c^2}}\right) \tag{20}$$

$$= \int_0^{v_f} m \left(1 - \frac{v^2}{c^2}\right)^{-3/2} vdv \tag{21}$$

$$= mc^2 \left(\frac{1}{1 - \frac{v_f^2}{c^2}} - 1 \right) \tag{22}$$

that we shall rewrite in the compact form:

$$K = mc^2(\gamma - 1) \tag{23}$$

1.3.2 For some authors as M. Planck and H.E. Ives, it is by no means clear the following Einstein's assertion (page 5):

> Thus it is clear that the difference $H - E$ can differ from the kinetic energy K of the body, with respect to the other system (ξ, η, ζ), only by an additive constant C, which depends on the choice of the arbitrary additive constants of the energies H and E. Thus we may place
>
> $$H_o - E_o = K_o + C \tag{24}$$
>
> $$H_1 - E_1 = K_o + C \tag{25}$$
>
> since C does not change during the emission of light.

1.3.3 Ives begins his argument on the circularity of Einstein's derivation of the mass energy relation by recalling Planck's objection to Einstein's assumption:

$$H - E = K + C \tag{26}$$

where E is observed form the object's proper frame (x, y, z) and H and K from the frame (ξ, η, ζ). He then develops Planck's objection by considering Einstein's equation (5):

$$(H_o - E_o) - (H_1 - E_1) = L(\gamma - 1) \tag{27}$$

derived in page 5 of Einstein's paper. Then, taking into account (22) and being m_o and m_1 respectively the masses of the body before and after the emission of the electromagnetic radiation,[2] he writes the corresponding kinetic energies K_o and K_1 as:

$$K_o = m_o c^2(\gamma - 1) \tag{28}$$

$$K_1 = m_1 c^2(\gamma - 1) \tag{29}$$

And therefore:

$$K_o - K_1 = (m_o - m_1)c^2(\gamma - 1) \tag{30}$$

$$(\gamma - 1) = \frac{K_o - K_1}{(m_o - m_1)c^2} \tag{31}$$

He now rewrites equation (27) as:

$$(H_o - E_o) - (H_1 - E_1) = L\frac{K_o - K_1}{(m_o - m_1)c^2} \tag{32}$$

that can be considered as the difference between:

$$H_o - E_o = \frac{L}{(m_o - m_1)c^2}(K_o + C) \tag{33}$$

$$H_1 - E_1 = \frac{L}{(m_o - m_1)c^2}(K_1 + C) \tag{34}$$

[2]The original notation was m and m'.

because in fact:

$$(H_o - E_o) - (H_1 - E_1) = \frac{L}{(m_o - m_1)c^2}[(K_o + C) - (K_1 + C))] \tag{35}$$

$$= \frac{L}{(m_o - m_1)c^2}(K_o - K_1) \tag{36}$$

1.3.4 Equations (33)-(34) differs from Einstein's (6) and (7):

$$H_o - E_o = K_o + C \tag{37}$$

$$H_1 - E_1 = K_1 + C \tag{38}$$

by the multiplying factor $L/(m_o - m_1)c^2$. Thus Einstein's equations (37) and (38) implies implicitly that:

$$\frac{L}{(m_o - m_1)c^2} = 1 \tag{39}$$

And then that:

$$L = (m_o - m_1)c^2 \tag{40}$$

which is just what Einstein claimed to have proved. Or in other words, to prove $L = (m_1 - m_0)c^2$, Einstein (implicitly) assumed that $L = (m_1 - m_0)c^2$. This is the circularity Ives discovered in Einstein' derivation of the mass-energy relation.

1.3.5 Unnecessary as it may seem, let us recall that an argument cannot be refuted by other independent argument. If two independent arguments lead to contradictory conclusions, then both arguments are making use of an inconsistent assumption. That said, we must recognize that if the relativistic expression of the kinetic energy is $mc^2(\gamma - 1)$, then Ives's argument is right and Einstein's did not in fact appropriately derived the mass-energy relation. If that were the case, M. Planck would be its true discoverer.

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