GÖDEL THEOREM AND THE FIRST LAW OF LOGIC

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1. Paradoxes and inconsistencies

1-1. As everyone knows, science rests on two fundamental assumptions:[1] the laws of logic. But while the second law (Principle of non-Contradiction) plays a capital role in the development of formal systems, the first one (Principle of Identity) seems to play no role at all.

1-2. The first law is usually stated as: '*something is what it is, and not what it is not*; or as $A = A$, and the like. From the propositional point of view, the first law states:

$$p \Rightarrow p \tag{1}$$

that reads: if p is true then p is true; where p is any declarative proposition. Alternatively we could also write:

$$\neg p \Rightarrow \neg p \tag{2}$$

that reads: if p is not true then p is not true. Inferences (1) and (2) translate the sense of identity to the world of propositions. From this propositional point of view, the first laws states: if something is true then it is true; if something is not true then it is not true. It seems impossible to find a more simple and affordable principle to found our rational knowledge of reality. How could a formal system ignore the essential character of this law?

1-3. The first law does not hold for propositions for which it holds:

$$p \Rightarrow \neg p \tag{3}$$

unless we admit $p \wedge \neg p$ is not a contradiction. In fact, if that were case we would have:

$$\begin{cases} p \Rightarrow p \\ p \Rightarrow \neg p \end{cases} \tag{4}$$

And then:

$$p \Rightarrow (p \wedge \neg p) \tag{5}$$

So that if $p \wedge \neg p$ is a contradiction then p is inconsistent.

1-4. A well know example of proposition for which it holds (3) is the Liar Paradox[2]:

$$\text{p: p is false} \tag{6}$$

It is quite clear that if p is true, then it is true what it states; and being what it states that it is false, then it is false:

$$p \Rightarrow \neg p \tag{7}$$

[1] Or three if we include the Excluded Middle Law.

[2] The liar paradox can be expressed in many different ways, even in the form of several circularly related propositions.

that reads: if p is true then p is false.

1-5. Other paradoxes, as Richard paradox [5] or Grelling-Nelson paradox [1], [4], share with the Liar paradox some suspicious characteristics:

1. They are self-referent.
2. They are negative sentences of the form: *This sentence is not* **x**. Where *not* **x** may be, for instance: not true (false); not autologic (heterologic); not non-richardian (richardian).
3. The predicate **x** is sensitive to the double negation: not (not true) = true; not (non-autologic) = autologic; not (non-richardian) = richardian.
4. The sentences are not empirically verifiable so we have to speculate on if they are, or not, **x**.
5. The speculation on if the sentence is, or is not, **x** leads to the (contradiction?) paradox because the self-reference activates the effects of the double negation when we ask on if it is not **x**.
6. And mainly: since inference (3) holds for all of them, all of them violate the first law of logic.

2. Gödel theorem and the first law of logic

2-1. As is well known, Gödel's first incompleteness theorem solved the metamathematical deficiency of Richard's paradox[3]. On the other hand, and as Richard's paradox, the formula involved in Gödel's theorem is also self-referent, negative and not empirically verifiable. And, as we will immediately see, it does not satisfy the first law of logic either.

2-2. Gödel proved in his first incompleteness theorem [2], [3], that there exist a formula G in his formal calculus P such that if G is P-demonstrable then $\neg G$ is also P-demonstrable. And, similarly, if $\neg G$ is demonstrable then G is also P-demonstrable. Thus, if P is consistent then G is undecidable.

2-3. Let us analyze Gödel conclusions from the perspective of the first law of logic (in what follows P-dem stands for P-demonstrable).

- In a consistent system P, if G is P-dem then $\neg G$ is not P-dem.
- G states that G is not P-dem.
- Gödel proved that if G is P-dem (which means that G is false) then $\neg G$ is also P-dem (which, by consistency, entails that G is not P-dem; i.e that G is true).

In consequence, by Gödel proof and by formal consistency we have: If G is false then G is true. In symbols:

$$\neg G \Rightarrow G \tag{8}$$

2-4. The same argument on $\neg G$ goes as follows:

- In a consistent system P, if $\neg G$ is P-dem then G is not P-dem.
- G states that G is not P-dem.
- Gödel proved that if $\neg G$ is P-dem (which, by consistency, entails that G is not P-demo, i.e. that G is true) then G is P-dem (which means that G is false).

[3]The inclusion of metamathematical statements in the formal system.

In consequence, by Gödel proof and formal consistency we have: If G is true then G is false. In symbols:

$$G \Rightarrow \neg G \tag{9}$$

2-5. Thus, Gödel formula is one that if it is true then it its false, and if it is false then it is true. It is a lucky break for all of us that such formulas can not be demonstrated in formal systems...

2-6. If G satisfies the first law of logic we can write:

$$G \Rightarrow G \tag{10}$$

On the other hand, and according to (9), we have:

$$G \Rightarrow \neg G \tag{11}$$

Consequently, we can write:

$$G \Rightarrow G \wedge \neg G \tag{12}$$

which is a contradiction. Thus, either G does not satisfy the first law of logic, or G is inconsistent

2-7. As in the cases of the Liar, Grelling-Nelson and Richard paradoxes, it holds:

$$\begin{cases} G \Rightarrow \neg G \\ \neg G \Rightarrow G \\ \neg(G \Rightarrow G) & \text{(The first law of logic does not hold)} \end{cases} \tag{13}$$

Note that the role played by the double negation in the case of Liar, Grelling-Nelson and Richard paradoxes, is played in the case of G by the consistency requirement: If G is P-dem then $\neg G$ is not P-dem; and is $\neg G$ is P-dem the G is not P-dem.

2-8. Let us end by recalling our initial questions:

1. Is not the law $p \Rightarrow p$ a fundamental law of logic?
2. Should not all formula and propositions of formal systems satisfy all logic fundamental laws?
3. Otherwise, what is the purpose of those fundamental law?

Referencias

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